Pushdown Automata (PDA)

Reading: Chapter 6
PDA - the automata for CFLs

- What is?
  - FA to Reg Lang, PDA is to CFL
- PDA == [ $\varepsilon$-NFA + “a stack” ]
- Why a stack?

Diagram:
- Input string
- $\varepsilon$-NFA
- Accept/reject
- A stack filled with “stack symbols”
Pushdown Automata - Definition

A PDA $P := (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$:
- $Q$: states of the $\varepsilon$-NFA
- $\Sigma$: input alphabet
- $\Gamma$: stack symbols
- $\delta$: transition function
- $q_0$: start state
- $Z_0$: Initial stack top symbol
- $F$: Final/accepting states
\( \delta : Q \times \Sigma \times \Gamma \Rightarrow Q \times \Gamma \)

\( \delta : \) The Transition Function

\( \delta(q,a,X) = \{(p,Y), \ldots\} \)

1. state transition from q to p
2. \( a \) is the next input symbol
3. \( X \) is the current stack top symbol
4. \( Y \) is the replacement for \( X \); it is in \( \Gamma^* \) (a string of stack symbols)

- i. Set \( Y = \varepsilon \) for: Pop(\( X \))
- ii. If \( Y = X \): stack top is unchanged
- iii. If \( Y = Z_1Z_2\ldots Z_k \): \( X \) is popped and is replaced by \( Y \) in reverse order (i.e., \( Z_1 \) will be the new stack top)

<table>
<thead>
<tr>
<th>( Y = ? )</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = \varepsilon )</td>
<td>Pop(( X ))</td>
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<tr>
<td>( Y = X )</td>
<td>Pop(( X )) Push(( X ))</td>
</tr>
<tr>
<td>( Y = Z_1Z_2\ldots Z_k )</td>
<td>Pop(( X )) Push(( Z_k )) Push(( Z_{k-1} )) ... Push(( Z_2 )) Push(( Z_1 ))</td>
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Example

Let $L_{wwr} = \{ww^R \mid w \text{ is in } (0+1)^*\}$

- CFG for $L_{wwr}$: $S \rightarrow 0S0 \mid 1S1 \mid \epsilon$
- PDA for $L_{wwr}$:
- $P := (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

$= (\{q_0, q_1, q_2\}, \{0,1\}, \{0,1,Z_0\}, \delta, q_0, Z_0, \{q_2\})$
Initial state of the PDA:

PDA for $L_{wwr}$

1. $\delta(q_0, 0, Z_0) = \{(q_0, 0Z_0)\}$
2. $\delta(q_0, 1, Z_0) = \{(q_0, 1Z_0)\}$

   First symbol push on stack

3. $\delta(q_0, 0, 0) = \{(q_0, 00)\}$
4. $\delta(q_0, 0, 1) = \{(q_0, 01)\}$
5. $\delta(q_0, 1, 0) = \{(q_0, 10)\}$
6. $\delta(q_0, 1, 1) = \{(q_0, 11)\}$

   Grow the stack by pushing new symbols on top of old (w-part)

7. $\delta(q_0, \varepsilon, 0) = \{(q_1, 0)\}$
8. $\delta(q_0, \varepsilon, 1) = \{(q_1, 1)\}$
9. $\delta(q_0, \varepsilon, Z_0) = \{(q_1, Z_0)\}$

   Switch to popping mode, nondeterministically (boundary between w and $w^R$)

10. $\delta(q_1, 0, 0) = \{(q_1, \varepsilon)\}$
11. $\delta(q_1, 1, 1) = \{(q_1, \varepsilon)\}$

   Shrink the stack by popping matching symbols ($w^R$-part)

12. $\delta(q_1, \varepsilon, Z_0) = \{(q_2, Z_0)\}$

   Enter acceptance state
PDA as a state diagram

\[ \delta(q_i, a, X) = \{(q_j, Y)\} \]
PDA for $L_{wwr}$: Transition Diagram

- **Grow stack**
  - $0, Z_0/Z_0$
  - $1, Z_0/1Z_0$
  - $0, 0/0$
  - $0, 1/01$
  - $1, 0/0$
  - $1, 1/11$

- **Pop stack for matching symbols**
  - $0, 0/\epsilon$
  - $1, 1/\epsilon$

- **Switch to popping mode**
  - $\epsilon, Z_0/Z_0$
  - $\epsilon, 0/0$
  - $\epsilon, 1/1$

- **Go to acceptance**
  - $\epsilon, Z_0/Z_0$

- **Symbols**
  - $\Sigma = \{0, 1\}$
  - $\Gamma = \{Z_0, 0, 1\}$
  - $Q = \{q_0, q_1, q_2\}$

This would be a non-deterministic PDA
Example 2: language of balanced paranthesis

\[ \Sigma = \{ (, ) \} \]
\[ \Gamma = \{ Z_0, ( \} \]
\[ Q = \{ q_0, q_1, q_2 \} \]

- Grow stack
- Pop stack for matching symbols
- Switch to popping mode
- Go to acceptance (by final state) when you see the stack bottom symbol
- To allow adjacent blocks of nested paranthesis
Example 2: language of balanced paranthesis (another design)

\[
\Sigma = \{ (, ) \} \\
\Gamma = \{ Z_0, ( \} \\
Q = \{ q_0, q_1 \}
\]
A PDA has a configuration at any given instance:

\((q,w,y)\)

- \(q\) - current state
- \(w\) - remainder of the input (i.e., unconsumed part)
- \(y\) - current stack contents as a string from top to bottom of stack

If \(\delta(q,a,X) = \{(p,A)\}\) is a transition, then the following are also true:

- \((q,a,X) \rightarrow (p,\varepsilon,A)\)
- \((q,aw,XB) \rightarrow (p,w,AB)\)

|--- sign is called a “turnstile notation” and represents one move

|---* sign represents a sequence of moves
How does the PDA for $L_{www}$ work on input “1111”?

All moves made by the non-deterministic PDA

Acceptance by final state:

= empty input AND final state
Principles about IDs

- **Theorem 1:** If for a PDA, 
  \((q, x, A) \rightarrow^{*} (p, y, B)\), then for any string 
  \(w \in \Sigma^*\) and \(\gamma \in \Gamma^*\), it is also true that:
  - \((q, x w, A \gamma) \rightarrow^{*} (p, y w, B \gamma)\)

- **Theorem 2:** If for a PDA, 
  \((q, x w, A) \rightarrow^{*} (p, y w, B)\), then it is also true that:
  - \((q, x, A) \rightarrow^{*} (p, y, B)\)
There are two types of PDAs that one can design: those that accept by final state or by empty stack.

Acceptance by…

- **PDAs that accept by final state:**
  - For a PDA P, the language accepted by P, denoted by $L(P)$ by **final state**, is:
    - $\{w \mid (q_0, w, Z_0) \rightarrow^* (q, \varepsilon, A) \}, \text{ s.t., } q \in F$ 

- **PDAs that accept by empty stack:**
  - For a PDA P, the language accepted by P, denoted by $N(P)$ by **empty stack**, is:
    - $\{w \mid (q_0, w, Z_0) \rightarrow^* (q, \varepsilon, \varepsilon) \}, \text{ for any } q \in Q$.

Q) Does a PDA that accepts by **empty stack** need any final state specified in the design?

Checklist:
- Input exhausted?
- Is the stack empty?
Example: L of balanced parenthesis

PDA that accepts by final state

\( P_F: \)

\[ (, Z_0 / ( Z_0 \]

\[ , ( / ( ( \]

\[ ), ( / \varepsilon \]

An equivalent PDA that accepts by empty stack

\( P_N: \)

\[ (, Z_0 / ( Z_0 \]

\[ , ( / ( ( \]

\[ ), ( / \varepsilon \]

\[ \varepsilon, Z_0 / \varepsilon \]

How will these two PDAs work on the input: ( ( ( ) ( ) ) ( )

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PDA for $L_{wwr}$: Proof of correctness

- **Theorem:** The PDA for $L_{wwr}$ accepts a string $x$ by final state if and only if $x$ is of the form $ww^R$.

- **Proof:**
  - **(if-part)** If the string is of the form $ww^R$ then there exists a sequence of IDs that leads to a final state:
    
    $$(q_0,ww^R,Z_0) \rightarrow^* (q_0,w^R,wZ_0) \rightarrow^* (q_1,w^R,wZ_0) \rightarrow^* (q_2,\varepsilon,Z_0)$$

  - **(only-if part)**
    - Proof by induction on $|x|$
PDAs accepting by final state and empty stack are equivalent

- $P_F \leq PDA \text{ accepting by final state}
  - P_F = (Q_F, \sum, \Gamma, \delta_F, q_0, Z_0, F)$

- $P_N \leq PDA \text{ accepting by empty stack}
  - P_N = (Q_N, \sum, \Gamma, \delta_N, q_0, Z_0)$

- **Theorem:**
  - $(P_N \Rightarrow P_F)$ For every $P_N$, there exists a $P_F$ s.t. $L(P_F) = L(P_N)$
  - $(P_F \Rightarrow P_N)$ For every $P_F$, there exists a $P_N$ s.t. $L(P_F) = L(P_N)$
How to convert an empty stack PDA into a final state PDA?

$P_N \Rightarrow P_F$ construction

- Whenever $P_N$’s stack becomes empty, make $P_F$ go to a final state without consuming any addition symbol.
- To detect empty stack in $P_N$: $P_F$ pushes a new stack symbol $X_0$ (not in $\Gamma$ of $P_N$) initially before simulating $P_N$.

$P_F = (Q_N \cup \{p_0,p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$
Example: Matching parenthesis "(" ")"

\[ P_N: \] ( \{ q_0 \}, \{ (, ) \}, \{ Z_0, Z_1 \}, \delta_N, q_0, Z_0 )

\[ \delta_N: \]
\[ \delta_N(q_0,(,Z_0) = \{ (q_0,Z_1Z_0) \} \]
\[ \delta_N(q_0,(,Z_1) = \{ (q_0, Z_1Z_1) \} \]
\[ \delta_N(q_0,),Z_1) = \{ (q_0, \epsilon) \} \]
\[ \delta_N(q_0, \epsilon,Z_0) = \{ (q_0, \epsilon) \} \]

\[ P_f: \] ( \{ p_0,q_0 ,pf \}, \{ (, ) \}, \{ X_0,Z_0,Z_1 \}, \delta_f, p_0, X_0 , p_f )

\[ \delta_f: \]
\[ \delta_f(p_0, \epsilon,X_0) = \{ (q_0,Z_0) \} \]
\[ \delta_f(q_0,(,Z_0) = \{ (q_0,Z_1Z_0) \} \]
\[ \delta_f(q_0,(,Z_1) = \{ (q_0, Z_1Z_1) \} \]
\[ \delta_f(q_0,),Z_1) = \{ (q_0, \epsilon) \} \]
\[ \delta_f(q_0, \epsilon,Z_0) = \{ (q_0, \epsilon) \} \]
\[ \delta_f(p_0, \epsilon,X_0) = \{ (p_f, X_0) \} \]

Accept by empty stack

Accept by final state
How to convert an final state PDA into an empty stack PDA?

**P_F**==>**P_N** construction

- Main idea:
  - Whenever **P_F** reaches a final state, just make an \( \varepsilon \)-transition into a new end state, clear out the stack and accept
  - Danger: What if **P_F** design is such that it clears the stack midway **without** entering a final state?
    - to address this, add a new start symbol \( X_0 \) (not in \( \Gamma \) of **P_F**)

\[
\text{P}_N = (Q \cup \{p_0, p_e\}, \sum, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)
\]

**P_N**:

- **New start**
- \( \varepsilon, X_0/Z_0X_0 \)

- **P_F**
Equivalence of PDAs and CFGs
CFGs == PDAs ==> CFLs
Converting CFG to PDA

Main idea: The PDA simulates the leftmost derivation on a given $w$, and upon consuming it fully it either arrives at acceptance (by empty stack) or non-acceptance.
Converting a CFG into a PDA

Main idea: The PDA simulates the leftmost derivation on a given \( w \), and upon consuming it fully it either arrives at acceptance (by empty stack) or non-acceptance.

Steps:
1. Push the right hand side of the production onto the stack, with leftmost symbol at the stack top
2. If stack top is the leftmost variable, then replace it by all its productions (each possible substitution will represent a distinct path taken by the non-deterministic PDA)
3. If stack top has a terminal symbol, and if it matches with the next symbol in the input string, then pop it

State is inconsequential (only one state is needed)
Formal construction of PDA from CFG

- **Given**: \( G = (V, T, P, S) \)
- **Output**: \( P_N = (\{q\}, T, V \cup T, \delta, q, S) \)

\[ \delta: \]

- For all \( A \in V \), add the following transition(s) in the PDA:
  \[ \delta(q, \varepsilon, A) = \{ (q, \alpha) | "A \Rightarrow \alpha" \in P \} \]

- For all \( a \in T \), add the following transition(s) in the PDA:
  \[ \delta(q, a, a) = \{ (q, \varepsilon) \} \]
Example: CFG to PDA

- $G = ( \{S,A\}, \{0,1\}, P, S)$
- $P$:
  - $S \Rightarrow AS | \varepsilon$
  - $A \Rightarrow 0A1 | A1 | 01$
- $\text{PDA} = (\{q\}, \{0,1\}, \{0,1,A,S\}, \delta, q, S)$
- $\delta$:
  - $\delta(q, \varepsilon, S) = \{(q, AS), (q, \varepsilon)\}$
  - $\delta(q, \varepsilon, A) = \{(q,0A1), (q,A1), (q,01)\}$
  - $\delta(q, 0, 0) = \{(q, \varepsilon)\}$
  - $\delta(q, 1, 1) = \{(q, \varepsilon)\}$

How will this new PDA work?

Let's simulate string $0011$
Simulating string 0011 on the new PDA ...

PDA ($\delta$):
\[
\begin{align*}
\delta(q, \varepsilon, S) &= \{ (q, AS), (q, \varepsilon) \} \\
\delta(q, \varepsilon, A) &= \{ (q,0A1), (q,A1), (q,01) \} \\
\delta(q, 0, 0) &= \{ (q, \varepsilon) \} \\
\delta(q, 1, 1) &= \{ (q, \varepsilon) \}
\end{align*}
\]

Leftmost deriv.:
\[
\begin{align*}
S &= \Rightarrow AS \\
& \Rightarrow 0A1S \\
& \Rightarrow 0011S \\
& \Rightarrow 0011
\end{align*}
\]

Stack moves (shows only the successful path):

Accept by empty stack
Proof of correctness for CFG ==> PDA construction

- **Claim**: A string is accepted by G iff it is accepted (by empty stack) by the PDA

- **Proof**:
  - *(only-if part)*
    - Prove by induction on the number of derivation steps
  - *(if part)*
    - If $(q, wx, S) \Rightarrow^* (q, x, B)$ then $S \Rightarrow_{lm}^* wB$
Converting a PDA into a CFG

- **Main idea:** Reverse engineer the productions from transitions

  If $\delta(q,a,Z) \Rightarrow (p, Y_1 Y_2 Y_3 \ldots Y_k)$:
  1. State is changed from $q$ to $p$;
  2. Terminal $a$ is consumed;
  3. Stack top symbol $Z$ is popped and replaced with a sequence of $k$ variables.

- **Action:** Create a grammar variable called “[qZp]” which includes the following production:
  - $[qZp] \Rightarrow a[pY_1q_1][q_1Y_2q_2][q_2Y_3q_3] \ldots [q_{k-1}Y_kq_k]$

- Proof discussion (in the book)
Example: Bracket matching

To avoid confusion, we will use $b=\text{"(" and } e=\text{")}$. 

Let $A=\{q_0Z_0q_0\}$

Let $B=\{q_0Z_1q_0\}$

If you were to directly write a CFG:

$$S \Rightarrow b \ S \ e \ S \mid \varepsilon$$
Two ways to build a CFG

- Build a PDA → Construct CFG from PDA (indirect)
- Derive CFG directly (direct)

Similarly...

- Derive a CFG → Construct PDA from CFG (indirect)
- Design a PDA directly (direct)
Deterministic PDAs
This PDA for $L_{wwr}$ is non-deterministic.

- **Grow stack**
  - $0, Z_0/0Z_0$
  - $1, Z_0/1Z_0$
  - $0, 0/0$
  - $0, 1/01$
  - $1, 0/10$
  - $1, 1/11$

- **Pop stack for matching symbols**
  - $0, 0/\varepsilon$
  - $1, 1/\varepsilon$

- **Switch to popping mode**
  - $\varepsilon, Z_0/Z_0$
  - $\varepsilon, 0/0$
  - $\varepsilon, 1/1$

- **Accepts by final state**
  - $\varepsilon, Z_0/Z_0$

**Why does it have to be non-deterministic?**

- To remove guessing, impose the user to insert c in the middle.
Example shows that: Nondeterministic PDAs $\neq$ D-PDAs

D-PDA for $L_{wcwr} = \{wcw^R \mid c \text{ is some special symbol not in } w\}$

Grow stack

- $0$, $Z_0/0Z_0$
- $1$, $Z_0/1Z_0$
- $0$, $0/00$
- $0$, $1/01$
- $1$, $0/10$
- $1$, $1/11$

Pop stack for matching symbols

- $0$, $0/\varepsilon$
- $1$, $1/\varepsilon$

Switch to popping mode

- $c$, $Z_0/Z_0$
- $c$, $0/0$
- $c$, $1/1$

Accepts by final state

Note:
- all transitions have become deterministic

$q_0 \xrightarrow{c, Z_0/Z_0} q_1 \xrightarrow{c, 0/0} q_1 \xrightarrow{c, 1/1} q_2 \xrightarrow{\varepsilon, Z_0/Z_0}$
Deterministic PDA: Definition

- A PDA is *deterministic* if and only if:
  1. \( \delta(q,a,X) \) has *at most one* member for any \( a \in \Sigma \cup \{\varepsilon\} \)

\[ \Rightarrow \] If \( \delta(q,a,X) \) is non-empty for some \( a \in \Sigma \), then \( \delta(q, \varepsilon, X) \) must be empty.
PDA vs DPDA vs Regular languages

- Regular languages
- D-PDA
- non-deterministic PDA

$L_{w_{cw}}$ and $L_{w_{ww}}$
Summary

- PDAs for CFLs and CFGs
  - Non-deterministic
  - Deterministic
- PDA acceptance types
  1. By final state
  2. By empty stack
- PDA
  - IDs, Transition diagram
- Equivalence of CFG and PDA
  - CFG => PDA construction
  - PDA => CFG construction