Consider the basic op-amp circuit shown below. The transfer function is found to be

\[ H(s) = -\frac{Z_f(s)}{Z_i(s)} \]

Problem 1. Given the modeling assumptions that the positive and negative input currents of the op-amp are zero, and that the voltages at the positive and negative inputs to the op-amp are equal, derive the transfer function above.

Problem 2. Modify the circuit to have two inputs, as shown below. Derive the relation between output and inputs. Determine values of \( R_1, R_2, R_f \) so that i) \( v_0 = -(v_1 + v_2) \); ii) \( v_0 = -(v_1 + 2v_2) \).
A non-inverting amplifier is shown below.

\[ V_i = V_+ = V_- \]
\[ V_- = \frac{R_2}{R_1 + R_2} V_0 = V_i \]

Hence
\[ V_0 = \left( 1 + \frac{R_1}{R_2} \right) V_i \]

If \( R_2 \to \infty \), then
\[ \frac{V_0}{V_i} = 1 \]
Problem 3. The circuit below has transfer function

\[ H(s) = -\frac{Z_f(s)}{Z_i(s)} = -\frac{sRC}{1} \]

and so is a differentiator, with \( v_0(t) = -RC \frac{dv_i}{dt} \).

The figure below (courtesy of A. Holzer) shows the circuit response (in red) to a square pulse input (in black). What must the RC time constant be to get the response shown?

RC high-pass filter as lossy differentiator.
Problem 4. Derive the transfer function for the circuit below, and show that it is an integrator. For the waveforms below, determine the RC time constant of the circuit.

The figure below (courtesy of A. Holzer) shows the circuit response to a square pulse input. What is the RC time constant for the circuit?

RC low-pass filter as lossy integrator.
Problem 5. Derive the transfer function for the circuit below. Verify that it is a low-pass filter. Using a 1 $\mu$F capacitor, design the filter to have a (-3 dB) cutoff frequency of 1000 rad/sec and a low-frequency gain of 20 dB.

\[ H(s) = \frac{Z_f(s)}{Z_i(s)} = -\frac{\frac{R_2}{sC}}{\frac{R_2}{R_1} + \frac{1}{sC}} = -\left(\frac{R_2}{R_1}\right)\left(\frac{1}{s + \frac{1}{R_2C}}\right) \]

Set the cutoff frequency,

\[ \omega_c = 1000 = \frac{1}{R_2C} \]

So

\[ R_2 = \frac{1}{\omega_c C} = \frac{1}{10^3 \times 10^{-6}} = 1,000 \ \Omega \]

The dc gain is

\[ |H(j0)| = \frac{R_2}{R_1} \]

Setting this to 20 dB (a gain of 10), then

\[ R_1 = \frac{R_2}{10} = 100 \ \Omega \]
Problem 6. Derive the transfer function for the circuit below. Verify that it is a high-pass filter. Using a 1 \( \mu F \) capacitor, design the filter to have a (-3 dB) cutoff frequency of 100 rad/sec and a high-frequency gain of 6 dB.

\[
H(s) = \frac{-Z_f(s)}{Z_i(s)} = -\frac{R_2}{R_1 + \frac{1}{sC}} = \left(-\frac{R_2}{R_1}\right)\left(\frac{s}{s + \frac{1}{R_1C}}\right)
\]

Set the cutoff frequency,
\[
\omega_c = 100 = \frac{1}{R_1C}
\]

So
\[
R_1 = \frac{1}{\omega_cC} = \frac{1}{10^210^{-6}} = 10,000 \; \Omega
\]

The high-frequency gain is
\[
|H(j\infty)| = \frac{R_2}{R_1}
\]

Setting this to 6 dB (a gain of 2), then
\[
R_2 = 2R_1 = 20,000 \; \Omega
\]
Problem 7. Design the active filter below to boost high frequencies more than low frequencies. Specifically, make the (asymptotic) Bode magnitude gain i) 6 dB for $0 \leq f \leq 100$ Hz, and ii) 26 dB for $f \geq 1000$ Hz. Let $C_1 = 1 \mu F$.

Solution: Sketch the asymptotic magnitude Bode plot. Find the circuit transfer function. Determine the low-frequency gain, the high-frequency gain, and the pole and zero break frequencies.
Problem 8. Determine the impedance of the series combination of resistor $R_1$ with the parallel combination of resistor $R_2$ and capacitor $C$.

\[
Z(s) = R_1 + \frac{R_2}{sR_2C_f + 1} = \frac{sR_1R_2C_f + R_1 + R_2}{sR_2C_f + 1}
\]

Now, let the input and feedback impedances be of the form above, with $R_a, R_b, C_i$ the parameters in the input impedance section, and determine the transfer function of the circuit below. Identify all finite poles and zeros.

\[
H(s) = -\frac{R_1}{R_a} \left( \frac{s + \frac{1}{R_bC_i}}{s + \frac{1}{R_{eq,f}C_f}} \right) \left( \frac{s + \frac{1}{R_{eq,i}C_i}}{s + \frac{1}{R_{eq,f}C_f}} \right)
\]

where $R_{eq,i} = \frac{R_a R_b}{R_a + R_b}$ and $R_{eq,f} = \frac{R_1 R_2}{R_1 + R_2}$. Note that all poles and zeros must be real valued. Also, the pole/zero pair contributed by the input impedance are ordered, with the pole (at $s = \frac{-1}{R_{eq,i}C_i}$) to the left of the zero (at $s = \frac{-1}{R_bC_i}$). This follows since $R_b \geq R_{eq,i}$. The pole/zero pair contributed by
the feedback impedance are ordered opposite, with the zero to the left of the pole. Hence, certain pole/zero orderings are not possible (for example, both zeros located to the right of both poles).

To clarify the situation, define the zero due to the input impedance as \( s = -\frac{1}{R_{bi}C_i} = -\omega_{z,i} \), the pole due to the input impedance as \( s = -\frac{1}{R_{ei}C_i} = -\omega_{p,i} \), the zero due to the feedback impedance as \( s = -\frac{1}{R_{eq,i}C_f} = -\omega_{z,f} \), and the pole due to the feedback impedance as \( s = -\frac{1}{R_{eq,f}C_f} = -\omega_{p,f} \). The pole/zero restrictions then imply that the pole/zero break frequencies must satisfy \( \omega_{p,i} \geq \omega_{z,i} \) and \( \omega_{z,f} \geq \omega_{p,f} \).

Now, for each of the pole-zero configurations shown, sketch the shape of the (asymptotic) magnitude Bode plot.

![Bode plot diagram]

Design the filter to approximate the frequency response shown below. What pole-zero configuration is required. Select component values. Use 10 nF capacitors. Plot the magnitude Bode frequency response for your design and compare to the plot below.

![Frequency response graph]
Second-Order Wideband Bandpass Active Filters

Basic approach: Use a cascade of two first-order filters, one highpass and the other lowpass. This is justified due to the high input impedance of an operational amplifier.

\[ H(s) = H_1(s)H_2(s) \]

Note that the bandpass filter upper cutoff frequency is the lowpass filter cutoff frequency, and the lower bandpass filter cutoff frequency is the highpass filter cutoff frequency. Hence, a reasonable design requires that the lowpass filter cutoff frequency be larger than the highpass filter cutoff frequency.

Problem 9. Design a bandpass filter for the audio range \([100, 10,000]\) kHz, to provide a 20 dB gain. Use \(0.2 \mu F\) capacitors.

Solution. Set the upper cutoff frequency as the lowpass filter cutoff frequency,

\[ \omega_{c2} = \omega_L = 2\pi \times 10,000. \]

Using the transfer function for a lowpass active filter (from problem 5 above)

\[ H(s) = -\frac{Z_f(s)}{Z_i(s)} = -\frac{R_2/SC}{R_2 + 1/SC} = -\left(\frac{R_2}{R_1}\right) \left(\frac{1}{s + 1/R_2C}\right) \]

Then \(\omega_{c2} = \frac{1}{R_2C}\), so

\[ R_2 = \frac{1}{2\pi \times 10,000 \times 0.2 \times 10^{-6}} = 79.6 \Omega. \]

Making the “dc” gain unity implies the input resistor is \(R_1 = R_2 = 79.6 \Omega\).

Using the transfer function for the highpass filter (from problem 6 above)

\[ H(s) = -\frac{Z_f(s)}{Z_i(s)} = -\frac{R_2}{R_1 + 1/SC} = -\left(\frac{R_2}{R_1}\right) \left(\frac{s}{s + 1/R_1C}\right) \]

Then the highpass cutoff frequency must be 100 Hz, so

\[ \omega_{c1} = \frac{1}{R_1C} \rightarrow R_1 = \frac{1}{2\pi \times 100 \times 0.2 \times 10^{-6}} = 7,958 \Omega. \]
Setting $R_2 = R_1 = 7,958 \Omega$ makes the highpass filter section unity gain. If we set

$$R_2 = 10 \times R_1 = 79.58 \, k\Omega$$

then the highpass filter section gain is 10 (20 dB), as required. The final design is the cascade of the lowpass and highpass sections, as shown below.

The overall transfer function is

$$H(s) = H_1(s)H_2(s) = \frac{R_2}{R_1} \left(\frac{s}{s + 200\pi}\right) \left(\frac{20,000\pi}{s + 20,000\pi}\right) = \frac{200,000\pi s}{s^2 + 20,200\pi s + 4\pi^2 \times 10^6}.$$  

The Bode plots are shown below. The bandwidth is about 10,000, as required.
Broadband Band-Reject Filters

To build a second-order band-reject filter, we can place in parallel (that is, add together) the transfer functions of a lowpass and a bandpass filter, i.e.,

\[ H(s) = H_1(s) + H_2(s). \]

In this case the lowpass filter cutoff frequency is the lower cutoff frequency of the band-reject filter, and the highpass filter cutoff frequency is the upper cutoff frequency of the band-reject filter, as shown below. A summing circuit (problem 2 above) is used to combine the lowpass and highpass filters.

![Diagram of Broadband Band-Reject Filters](image)

A circuit diagram is shown below.
The top-left circuit is a lowpass filter, where, for simplicity the two resistor values are set equal.

The transfer function is then (from problem 5)

\[ H_L(s) = - \frac{1}{\frac{1}{R_L C_L} + s} = -\frac{1}{\frac{1}{R_L C_L} + s}. \]

The lower-left circuit is a one-pole highpass filter, also with the two resistors set equal, and hence (from problem 6) with transfer function

\[ H_L(s) = -\frac{s}{s + \frac{1}{R_H C_H}}. \]

Setting \( R_1 = R_2 \) in the adder portion of the circuit, the overall transfer function is then

\[ H(s) = \frac{R_f}{R_1} (H_1(s) + H_2(s)). \]

Setting the lowpass filter cutoff frequency to the lower cutoff frequency of the band-reject filter, let

\[ \omega_c_1 = \frac{1}{R_L C_L}. \]

Similarly, set the highpass filter cutoff frequency to the band-reject filter upper cutoff frequency, so

\[ \omega_c_2 = \frac{1}{R_H C_H}. \]

The overall transfer function is then

\[ H(s) = \frac{R_f}{R_1} (H_1(s) + H_2(s)) = \frac{R_f}{R_1} \left( \frac{s^2 + 2\omega_c_1 s + \omega_c_1 \omega_c_2}{s^2 + (\omega_c_1 + \omega_c_2) s + \omega_c_1 \omega_c_2} \right). \]

As \( s \to 0 \) and as \( s \to \infty \) the overall filter gain goes to \( K = \frac{R_f}{R_1} \). The center frequency is

\[ \omega_0 = \sqrt{\omega_c_1 \omega_c_2}. \]
Problem 10. Design a wideband band-reject filter with \( \omega_c = 2\pi \times 100 \text{ rad/s} \) (100 Hz) and 
\( \omega_c = 2\pi \times 10,000 \text{ rad/s} \) (10,000 Hz) and unity gain at low and high frequencies. (This rejects
the middle of the audible frequency range.) Use 0.2 \( \mu F \) capacitors. Plot the frequency response.

What is the gain at the center frequency?

Solution. Set \( R_f = R_1 \) to get unity gain at high and low frequencies. Then
\[
\omega_c = \frac{1}{R_L C_L} = 2\pi \times 100 \quad \Rightarrow \quad R_L = \frac{1}{2\pi \times 100 \times 0.2 \times 10^{-6}} = 15.9 \text{ k}\Omega.
\]
\[
\omega_c = \frac{1}{R_H C_H} = 2\pi \times 10,000 \quad \Rightarrow \quad R_H = \frac{1}{2\pi \times 10,000 \times 0.2 \times 10^{-6}} = 159 \Omega.
\]
The center frequency is
\[
\omega_0 = \sqrt{\omega_c \omega_c} = 2\pi \times 1000,
\]
and the transfer function is
\[
H(s) = \frac{R_f}{R_1} \left( \frac{s^2 + 2\omega_c s + \omega_c \omega_c}{s^2 + (\omega_c + \omega_c)s + \omega_c \omega_c} \right) = \frac{s^2 + 400\pi s + (2000\pi)^2}{s^2 + 2\pi(10,100)s + (2000\pi)^2}.
\]
The Bode plots are shown below. The gain at the center frequency is – 34 dB.

```
>> sys=tf([1 400*pi (2000*pi)^2],[1 2*pi*10100 (2000*pi)^2]);
>> bode(sys)
```
Magnitude and Frequency Scaling

Magnitude scaling: increase impedance by a factor $k_m$ at (each) fixed frequency.

\[ L' = k_m L, \quad R' = k_m R \]
\[ C' = \frac{C}{k_m} \]

since
\[ j\omega L' = k_m \times j\omega L \]
\[ \frac{1}{j\omega C'} = \frac{1}{k_m} \times j\omega C \]

Frequency scaling: Maintain impedance at frequency scaled by $k_f$.

\[ L' = \frac{L}{k_f}, \quad R' = R \]
\[ C' = \frac{C}{k_f} \]

since
\[ j\omega L' = \frac{j\omega L}{k_f} \]
\[ \frac{1}{j\omega C'} = \frac{1}{k_f} \times j\omega C \]

Combining magnitude and frequency scaling:

\[ R' = k_m R \]
\[ L' = \frac{k_m L}{k_f} \]
\[ C' = \frac{C}{k_m k_f} \]

Example: A 2\textsuperscript{nd}-order series RLC bandpass filter with normalized center frequency $\omega_0 = 1$ rad/s, bandwidth $\beta = 1/2$ rad/s and quality factor $Q = 2$, has parameter values $L = 1$ H, $C = 1$ F, and $R = \frac{1}{2}$ $\Omega$. Use magnitude and frequency scaling to design a 2\textsuperscript{nd}-order bandpass filter with quality factor 2 and center frequency 10,000 rad/s. Use a 1 k$\Omega$ resistor.

Solution: Use $R' = k_m R$ to solve for the magnitude scale factor

\[ k_m = \frac{R'}{R} = \frac{1,000}{1/2} = 2,000. \]

The frequency scaling factor is

\[ k_f = \frac{\omega_0'}{\omega_0} = \frac{10,000}{1} = 10,000. \]

The new design is then $R' = 1,000$ $\Omega$,
\[ L' = \frac{k_m L}{k_f} = \frac{2,000 \times 1}{10,000} = 0.2 \text{ H} \]
\[ C' = \frac{C}{k_m k_f} = \frac{1}{2,000 \times 10,000} = 50 \text{ nF} \]

As a check, the designed center frequency is
\[ \omega_0' = \frac{1}{\sqrt{L' C'}} = \frac{1}{\sqrt{0.2 \times 5 \times 10^{-8}}} = 10,000, \]
and the bandwidth is $\beta = \frac{R'}{L'} = \frac{1,000}{0.2} = 5,000$, so the quality factor is $Q = \frac{10,000}{5,000} = 2$. 
Problem 11. Consider the active circuit shown below. Unlike most of the previous circuits considered, in this circuit the positive op amp input is not grounded. Determine the filter transfer function.

The circuit is a second-order lowpass filter with transfer function is

\[ H(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{s^2 + \frac{2}{R C_1} s + \frac{1}{R^2 C_1 C_2}}. \]

Now, consider the normalized design where we set \( R = 1 \Omega, C_1 C_2 = 1 \), and define \( b_1 = \frac{2}{C_1} \). Then the transfer function becomes

\[ H(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{s^2 + b_1 s + 1}. \]

Since the value of \( b_1 \) can be freely adjusted, the poles of \( H(s) \) can be complex-valued (unlike the active circuits discussed above). Setting \( \omega = 1 \), the frequency response magnitude,

\[ |H(j \omega)| = \frac{1}{|1 - \omega^2 + j b_1 \omega|}, \]

becomes \( |H(j 1)| = \frac{1}{|jb_1|} = \frac{1}{b_1} \). The figure below shows the filter frequency response magnitude (in dB) for various values of \( b_1 \).
Design the active filter to be a second-order Butterworth lowpass filter with cutoff frequency 1 rad/s. Then, use magnitude and frequency scaling to design a second-order Butterworth filter with cutoff frequency 1,000 rad/sec. Use 10 kΩ resistors.

Solution: Since the cutoff frequency is $\omega_0 = 1$ rad/s, and the second-order lowpass Butterworth transfer function is

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{\omega_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2},$$

it follows that $b_1 = \sqrt{2}$. Since $b_1 = \frac{2}{c_1}$, then $C_1 = \frac{2}{b_1} = \frac{2}{\sqrt{2}} = \sqrt{2}$. Finally, using $C_1C_2 = 1$, then $C_2 = \frac{1}{c_1} = \frac{1}{\sqrt{2}} \text{ F}$.

The normalized (to cutoff frequency 1 rad/s) design then has parameter values $R = 1 \Omega$, $C_1 = \sqrt{2} \text{ F}$ and $C_2 = \frac{1}{\sqrt{2}} \text{ F}$.

The frequency and magnitude scaling factors are found as follows.

$$k_f = \frac{\omega_{0,new}}{\omega_{0,old}} = \frac{1,000}{1} = 1,000$$

Since $R' = k_m R$, then

$$k_m = \frac{R'}{R} = \frac{10,000}{1} = 10,000$$

Then the scaled capacitor values are

$$C_1' = \frac{C_1}{k_f k_m} = \sqrt{2} \times 10^{-7} \text{ F}$$

and

$$C_1' = \frac{C_1}{k_f k_m} = \frac{1}{\sqrt{2}} \times 10^{-7} \text{ F}.$$
As a check on the design, from the transfer function on the previous page, the numerator must be the cutoff frequency squared. Evaluating using the scaled component values,

\[
\frac{1}{R^2C_1C_2} = \frac{1}{(10^4)^2 \times \sqrt{2} \times 10^{-7} \times \frac{1}{\sqrt{2}} \times 10^{-7}} = 10^6 = \omega_0^2
\]
as required.
Problem 12. The circuit in Problem 11 is modified by transposing resistors and capacitors to form the second-order highpass active filter shown below.

![Second-order highpass active filter](image)

The transfer function is

\[ H(s) = \frac{V_0(s)}{V_i(s)} = \frac{s^2}{s^2 + \frac{2}{R_2C}s + \frac{1}{C^2R_1R_2}}. \]

Let \( C = 1 \, \text{F}, \ R_1R_2 = 1 \) and define \( b_1 = \frac{2}{R_2} \). The transfer function becomes

\[ H(s) = \frac{V_0(s)}{V_i(s)} = \frac{s^2}{s^2 + b_1s + 1}. \]

This is a normalized highpass filter with cutoff frequency \( \omega_0 = 1 \). The value of \( b_1 \) can be freely adjusted, so that the poles can be designed to be either real-valued or complex valued. With the choice \( b_1 = \sqrt{2} \), the filter is a normalized second-order Butterworth highpass filter.

**Design Problem:** Design a second-order active highpass Butterworth filter to have cutoff frequency 10,000 rad/s. Use 10 nF capacitors. Specify the component values and then verify that your solution is correct by comparing \( \frac{1}{C^2R_1R_2} \) to \( \omega_0^2 \), and comparing \( \frac{2}{R_2C} \) to \( \sqrt{2}\omega_0 \).

**Solution.** The normalized highpass filter must have transfer function

\[ H(s) = \frac{s^2}{s^2 + \sqrt{2}s + 1}. \]

Comparing to the transfer function above, we must select \( b_1 = \sqrt{2} \). Since \( b_1 = \frac{2}{R_2} \) it follows that \( R_2 = \frac{2}{b_1} = \sqrt{2} \approx 1.414 \). Then, using \( R_1R_2 = 1 \), the final resistor value is \( R_1 = \frac{1}{R_2} \approx 0.707 \).

The frequency scale factor is selected as \( k_f = \frac{\omega'_0}{\omega_0} = \frac{10,000}{1} = 10,000 \). Using this value, and the value of capacitance for the final design (10 nF), the magnitude scale factor is found as

\[ C' = \frac{C}{k_f k_m} \Rightarrow k_m = \frac{C}{k_f C'} = \frac{1}{10,000 \times 10^{-8}} = 10,000. \]

Finally, \( R'_1 = k_m R_1 = 7.07 \, k\Omega \) and \( R'_2 = k_m R_2 = 14.14 \, k\Omega \).
Problem 12. Higher-Order Butterworth Filter Design

Table 15.1 in the text provides normalized Butterworth polynomials, in factored form, for orders 1 to 10. An Nth-order, normalized (to cutoff frequency 1 rad/s), Butterworth filter has transfer function

\[ H(s) = \frac{1}{D(s)} \]

where \( D(s) \) is the Nth-order Butterworth polynomial. The first few polynomials are listed below.

<table>
<thead>
<tr>
<th>Order, ( N )</th>
<th>( D(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( s^2 + \sqrt{2}s + 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( s^3 + 2s^2 + 2s + 1 = (s + 1)(s^2 + s + 1) )</td>
</tr>
<tr>
<td>4</td>
<td>( (s^2 + 0.765s + 1)(s^2 + 1.848s + 1) )</td>
</tr>
<tr>
<td>5</td>
<td>( (s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1) )</td>
</tr>
</tbody>
</table>

A higher-order Butterworth filter can be realized as a cascade of first- and second-order active filter sections. The design strategy is then summarized as follows.

i) Determine the required filter order, \( N \).

ii) If \( N \) is even, then the filter is realized as a cascade of \( N/2 \) second-order sections. If \( N \) is odd, then the realization uses one first-order active filter section, cascaded with \( (N-1)/2 \) second-order sections.

iii) First design the filter for a normalized cutoff frequency of 1 rad/s using the Butterworth polynomials above and the generic second-order active filter described in problem 10 (and, if a first-order section is required, using the first-order low-pass filter described in problem 5).

iv) Use magnitude and frequency scaling to modify the design to have the required cutoff frequency.

Example. Design a Butterworth lowpass filter to satisfy the following requirements.

a) The gain is no less than -1 dB for frequencies below 2 kHz;

b) The gain is no larger than -20 dB for frequencies above 6 kHz.

Solution. The minimum required filter order is \( N = 3 \). The -3 dB cutoff frequency can then be selected as any value satisfying \( 15,740 \leq \omega_0 \leq 17,528 \). For convenience, the value \( \omega_0 = 16,000 \) rad/s is selected (2,747 kHz). The normalized filter is then realized as a cascade of one 1st-order section and one 2nd-order section, using the 3rd-order Butterworth polynomial in the table above. The overall transfer function is then realized as

\[ H(s) = H_1(s)H_2(s) \]

First-order section. The desired 1st-order transfer function is \( H_1(s) = \frac{1}{s+1} \). Using problem 5 above, a first-order lowpass filter can be realized as an active filter by selecting the input impedance to be resistor \( R \), and selecting the feedback impedance as the parallel combination of resistor \( R \) and capacitor \( C \), as shown in the circuit below. The transfer function is then
To design the normalized filter, simply set \( R_1 = R_2 = 1 \Omega \) and \( C = 1 \) F. The resulting transfer function is then

\[
H_1(s) = \frac{1}{s + 1}.
\]

The 2nd-order section is designed using the second factor of the third-order Butterworth polynomial, so the transfer function is \( H_2(s) = \frac{1}{s^2 + s + 1} \). Using the active filter circuit from problem 10, shown again below, the circuit transfer function is

\[
H(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{s^2 + \frac{2}{R C_1} s + \frac{1}{R^2 C_1 C_2}} = \frac{1}{s^2 + b_1 s + 1}
\]

for the choice \( R = 1 \Omega, C_1 C_2 = 1 \), and letting \( b_1 = \frac{2}{C_1} \). To design the required transfer function, we simply select \( b_1 = 1 \), implying \( C_1 = 2 \) and hence that \( C_2 = 0.5 \) (since we must have \( C_1 C_2 = 1 \)). The normalized design is then \( R = 1 \Omega, C_1 = 2 \) F, \( C_2 = 0.5 \) F.
To complete the design, magnitude and frequency scaling are used to select the final resistor and capacitor values. The frequency scale factor is determined as

$$k_f = \frac{\omega'_0}{\omega_0} = \frac{16000}{1} = 16,000.$$  

To select the magnitude scale factor, we can select some suitable value of resistor to use. For example, suppose that we select $R' = 10 \, k\Omega$. Then the magnitude scale factor can be chosen as

$$k_f = \frac{R'}{R} = \frac{10,000}{1} = 10,000.$$  

The capacitor values are then

$$C' = \frac{C}{k_f k_m}.$$  

The final capacitor values are $C = 6.25 \, nF$ for the 1st-order section and $C_1 = 12.5 \, nF$, $C_2 = 3.125 \, nF$ for the 2nd-order section.

The final transfer function is

$$H(s) = -\frac{\omega_0^3}{s^3 + 2\omega_0 s^2 + 2\omega_0^2 s + \omega_0^3}$$

where $\omega_0 = 4000\pi \, \text{rad/s} \,(2 \, \text{kHz})$ is the -3dB cutoff frequency. Note that the filter is inverting (because the first-order section is inverting, but the second-order section is not). The final circuit is shown below.
Problem: Using 10 nF capacitors, design a 4th-order Butterworth highpass filter with cutoff frequency 1,000 Hz.

Solution. The normalized transfer function (for cutoff frequency 1 rad/s) for a 4th-order Butterworth highpass filter is (using the 4th-order Butterworth polynomial from the table above)

\[ H(s) = \frac{s^4}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}. \]

The filter is realized as the cascade of two 2nd-order sections. The first has transfer function

\[ H_1(s) = \frac{s^2}{(s^2 + 0.765s + 1)}. \]

From the 2nd-order highpass filter example above, the normalized capacitor values are \( C = 1 \) F. We select \( b_1 = 0.765 \). Then, since \( b_1 = \frac{2}{R_2} \) it follows that \( R_2 = \frac{2}{b_1} = \frac{2}{0.765} = 2.614 \Omega \). Using \( R_1 R_2 = 1 \), the final resistor value is \( R_1 = \frac{1}{R_2} \approx 0.328 \Omega \).

The second section has transfer function

\[ H_2(s) = \frac{s^2}{(s^2 + 1.848s + 1)}. \]

In this case, \( b_1 = 1.848 \), so that \( R_2 = \frac{2}{b_1} = \frac{2}{1.848} = 1.082 \Omega \) and \( R_1 = \frac{1}{R_2} = 0.924 \Omega \).

The frequency scale factor is

\[ k_f = \frac{\omega'_0}{\omega_0} = \frac{2\pi \times 1000}{1} = 2000\pi. \]

The magnitude scale factor is

\[ k_m = \frac{C}{k_f C^7} = \frac{1}{2000\pi \times 10^{-8}} = 15,915. \]

The final capacitor values are all 10 nF, and the final resistor values are \( R_1 = 5.22 \) k\( \Omega \) and \( R_2 = 41.6 \) k\( \Omega \) for the first section, and \( R_1 = 14.7 \) k\( \Omega \) and \( R_2 = 17.22 \) k\( \Omega \) for the second section.
The final circuit is shown below for the 4th-order highpass Butterworth filter.

\[ R_1 = 5.22 \text{ k}\Omega \]
\[ R_2 = 41.6 \text{ k}\Omega \]
\[ R_1 = 14.7 \text{ k}\Omega \]
\[ R_2 = 17.22 \text{ k}\Omega \]

\[ C = 10 \text{ nF} \]

\[ v_i(t) \]
\[ v_o(t) \]