Recall the simple RC circuit shown below.

\[ v(t) \]

\[ R \]

\[ C \]

\[ v(t) \]

The transfer function is \( H(s) = \frac{1}{RCs + \frac{1}{RC}} \), so the system has one pole at \( s = -\frac{1}{RC} \). The unit-step response is

\[ v_0(t) = \left(1 - e^{-\frac{t}{RC}}\right)u(t) \]

The 10-90% rise time is \( t_r \approx 2.2RC \) sec.

Consider the RLC circuit below.

\[ v(t) \]

\[ R \]

\[ L \]

\[ C \]

\[ v(t) \]

The transfer function is \( H(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \), with poles \( s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \).

Assume the case that the poles are complex-valued, and let

\[ H(s) = \frac{1}{LC} \frac{\omega_n^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \]

where \( \omega_n = \frac{1}{\sqrt{LC}} \) is called the natural frequency and \( \xi = \frac{R}{2\sqrt{L}} \) is called the damping factor. The poles of \( H(s) \) are complex valued, at location

\[ s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\xi\omega_n \pm j\omega_n \sqrt{1-\xi^2} = -\xi\omega_n \pm j\omega_n. \]

The poles then lie in the left-half s-plane on a circle of radius \( \omega_n \), at an angle of \( \pm \beta \) from the negative real axis, where \( \beta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \).
Example. Below is the pole-zero plot for the case $\xi = 0.8$ and $\omega_n = 1$.

![Pole-Zero Map](image)

**Unit-Step Response**
The unit-step response is found by setting $V_i(s) = 1/s$ and finding the inverse Laplace transform of $V_0(s) = \frac{\omega_n^2}{s(s + \xi \omega_n - j \omega_d)(s + \xi \omega_n + j \omega_d)}$. Using partial fraction expansion,

$$V_0(s) = \frac{K_1}{s} + \frac{K_2}{s + \xi \omega_n - j \omega_d} + \frac{K_3}{s + \xi \omega_n + j \omega_d},$$

where $K_1 = 1$, $K_2 = -\frac{1}{2} + j \frac{\xi}{2\sqrt{1 - \xi^2}}$, and $K_3 = K_2^*$. The unit-step response is then found to be

$$v_0(t) = \left[1 - e^{-\xi \omega_n t} \left(\cos(\omega_d t) + \frac{\xi}{\sqrt{1 - \xi^2}} \sin(\omega_d t)\right)\right] u(t),$$

or

$$v_0(t) = \left[1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1 - \xi^2}} \left(\sin \left[\omega_d t + \tan^{-1} \left(\frac{\sqrt{1 - \xi^2}}{\xi}\right)\right]\right)\right] u(t).$$
The 100% rise time, peak time, maximum overshoot, and 2% settling time are summarized as follows.

100% Rise time: \( t_r = \frac{\pi - \beta}{\omega_d} \), where \( \beta = \tan^{-1}\left(\frac{\sqrt{1 - \xi^2}}{\xi}\right) \), and \( \omega_d = \omega_n \sqrt{1 - \xi^2} \).

Peak time: \( t_p = \frac{\pi}{\omega_d} \).

Maximum overshoot: \( M_p = e^{\frac{-\xi}{\sqrt{1-\xi^2}}} \).

2% Settling time: \( t_s = \frac{4}{\xi \omega_n} \).

Matlab code for generating the figure above is presented below.

```matlab
>> t=[0:0.01:10];dp=0.3;wn=1.0;
>> wd=wn*sqrt(1-dp^2);
>> v1=1-exp(-dp*wn*t).*(cos(wd*t)+(dp/sqrt(1-dp^2))*sin(wd*t));
>> dp=0.5;
>> wd=wn*sqrt(1-dp^2);
>> v2=1-exp(-dp*wn*t).*(cos(wd*t)+(dp/sqrt(1-dp^2))*sin(wd*t));
>> dp=0.707;
>> wd=wn*sqrt(1-dp^2);
>> v3=1-exp(-dp*wn*t).*(cos(wd*t)+(dp/sqrt(1-dp^2))*sin(wd*t));
>> figure(1)
>> plot(t,v1,t,v2,'-t',v3,'--')
>> xlabel('Time, t, sec')
>> ylabel('Unit-Step Response')
>> title('2nd-Order System Step Response, \omega_n = 1, \xi = 0.3, 0.5, 0.707')
```
**Design Problem** Design a second-order RLC filter to have a 2% settling time of 1 μs, and an overshoot of no more than 5%. Use state variables and Matlab to plot the unit-step response to verify your design. Also use Matlab to plot the filter frequency response.

**Solution.** From the overshoot of 5%, solve for the damping constant.

\[ M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \Rightarrow \ln(M_p) = \frac{-\pi\xi}{\sqrt{1-\xi^2}} \Rightarrow \xi = \sqrt{\frac{-1}{\pi} \ln(M_p)} \frac{2}{1 + \left(\frac{-1}{\pi} \ln(M_p)\right)^2} \]

so for \( M_p = 0.05 \), we get \( \xi = 0.6987 \) .... Rounding up to \( \xi = 0.7 \), then the actual overshoot is \( M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 0.04599 \), or about 4.6%. To make the 2% settling time equal 1 μs, set : \( t_s = \frac{4}{\xi\omega_n} = 10^{-6} \), and solve for \( \omega_n = \frac{4}{0.7 \times 10^{-6}} = 5.7143 \times 10^6 \text{ rad/s} \).

Suppose we select a 10 pF capacitor. Then the required inductance value is \( L = \frac{1}{\omega_n^2 C} = 0.00306 \text{ H} \). Finally, the resistor value is selected using \( \xi = \frac{R}{2\sqrt{\frac{L}{C}}} \) so

\[ R = 2\xi \sqrt{\frac{L}{C}} = 7.74 \text{ M}\Omega \]

The frequency response is plotted using Matlab, as follows, with the plot below. Note that the filter is very close to being a second-order Butterworth filter. (What makes it different?)

```matlab
>> wn=5.7143e6;damp=0.7;
>> sys=tf([wn^2],[1 2*damp*wn wn^2]);
>> bode(sys)
```

![Bode Diagram](image-url)
Frequency Response
The frequency response of the second-order system is evaluated as follows.

\[ H(s) \big|_{s=j\omega} = \frac{\sqrt{LC}}{s^2 + \frac{R}{L} s + \frac{1}{LC}} \big|_{s=j\omega} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \big|_{s=j\omega} = \frac{1}{1 - \frac{\omega_0^2}{\omega_n^2} + j \frac{2\xi\omega}{\omega_n}}. \]

The magnitude, in dB, is

\[ |H(j\omega)| = -20\log_{10} \left| 1 - \frac{\omega_0^2}{\omega_n^2} + j \frac{2\xi\omega}{\omega_n} \right| = -10\log_{10} \left[ \left( 1 - \frac{\omega_0^2}{\omega_n^2} \right)^2 + \left( \frac{2\xi\omega}{\omega_n} \right)^2 \right]. \]

Clearly, \(|H(j0)| = 0 \text{ dB} \) (unity “dc gain”). For \( \omega >> \omega_n \), the frequency response decreases at a rate of 40 dB per decade change in \( \omega \).

At \( \omega = \omega_n \), \(|H(j\omega)| = -20\log_{10}[2\xi] \text{ dB} \), so the gain at \( \omega = \omega_n \) can be either positive, zero, or negative (in decibels) depending on the value of damping constant.

The frequency response magnitude is shown below, for the normalized natural frequency \( \omega_n = 1 \), along with the Matlab code used to generate the figure. Note in the figure (middle curve) that the gain is unity (0 dB) at \( \omega = 1 \text{ rad/sec} \) for the case of damping constant \( \xi = 0.5 \). For \( 0 \leq \xi < \frac{1}{\sqrt{2}} \), the frequency response can be larger than unity (larger than 0 dB) for \( \omega \) in the neighborhood of the resonance frequency, \( \omega_n \). When the damping constant is small, this gain can be quite large.
function second_order_freq_res
% Plots the frequency response magnitude of
% a second order system with \( \omega_n = 1 \) and
% damping constant values 0.1, 0.3, 0.5, 0.7 and 0.9.
damp=[0.1 0.3 0.5 0.7 0.9]; % values of damping constant
w=[0.1:0.01:10]; % Range of frequencies
logw=log10(w);
figure(1)
hold on
for i=1:5
    H=freqs([1],[1 2*damp(i) 1],w);
    plot(logw,20*log10(abs(H)))
end
xlabel('Frequency, \( \omega \), rad/sec')
ylabel('|H(j\omega)|, dB')
title('Second-Order System Frequency Response Magnitude, \( \omega_n = 1 \)')
hold off

A few special cases.

\[
H(s) = \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + \frac{\omega_n^2}{\xi^2} s + \frac{\omega_n^2}{\xi^2}}
\]

1. Damping constant \( \xi = 1 \) (critically damped): repeated real-valued poles at \( s = -\omega_n \).
2. Damping constant \( \xi = 0.5 \), complex-valued poles at angle \( \pm 60^\circ \) with respect to the
   negative real axis. Maximum percent overshoot is 16.3 \%.
3. Damping constant \( \xi = \frac{1}{\sqrt{2}} \), complex-valued poles at angle \( \pm 45^\circ \) with respect to the
   negative real axis. Maximum percent overshoot is 4.32 \%. This is the case for a 2\textsuperscript{nd}-
   order lowpass Butterworth filter.