State variable representation for an $n$th-order linear differential equation with no derivatives of the input. Let the differential equation be of the following form:

$$\frac{d^n v}{dt^n} + a_{n-1} \frac{d^{n-1} v}{dt^{n-1}} + \cdots + a_1 \frac{dv}{dt} + a_0 v = b_0 g,$$

where $g(t)$ is the input and $v(t)$ is the unknown signal to be found. Define the $n$ state variables $x_1, x_2, \cdots, x_n$ as $x_1 = v$, $x_2 = \frac{dv}{dt}$, $\cdots$, $x_n = \frac{d^{n-1} v}{dt^{n-1}}$. The state variable representation is

$$\dot{x} = Ax + Bg,$$

$$y = Cx + Dg,$$

where $x = (x_1, \cdots, x_n)^T$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad D = [0].$$

The variable $y$ is the output, and for the equation above is simply the signal $v(t)$. In a circuit application, there can be multiple “outputs,” selected as voltages or currents of interest in the circuit.

State variable representation for an $n$th-order linear differential equation with derivatives of the input. Let the differential equation be of the following form:

$$\frac{d^n v}{dt^n} + a_{n-1} \frac{d^{n-1} v}{dt^{n-1}} + \cdots + a_1 \frac{dv}{dt} + a_0 v = b_0 \frac{d^n g}{dt^n} + b_{n-1} \frac{d^{n-1} g}{dt^{n-1}} + \cdots + b_1 \frac{dg}{dt} + b_0 g,$$

where $g(t)$ is the input and $v(t)$ is the unknown signal to be found. Define the $n$ state variables $x_1, x_2, \cdots, x_n$ as $x_1 = v - \beta_0 g$, $x_2 = \frac{dv}{dt} - \beta_0 \frac{dg}{dt} - \beta_1 g$, $\cdots$, $x_n = \frac{d^{n-1} v}{dt^{n-1}} - \beta_0 \frac{d^{n-1} g}{dt^{n-1}} - \cdots - \beta_{n-2} \frac{dg}{dt} - \beta_{n-1} g$.

where the values of $\beta_0, \cdots, \beta_n$ are determined from

$$\beta_0 = b_n, \beta_1 = b_{n-1} - a_{n-1} \beta_0, \beta_2 = b_{n-2} - a_{n-2} \beta_1 - a_{n-2} \beta_0, \cdots, \beta_n = b_0 - a_{n-1} \beta_{n-1} - \cdots - a_1 \beta_1 - a_0 \beta_0.$$

The state-variable representation is then (1)-(2), where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad D = [\beta_0].$$

Example 1. Consider the 2nd-order differential equation (from the circuit example in class lecture)
\[
\frac{d^2 v}{dt^2} + 16000 \frac{dv}{dt} + 10^8 v = 0.
\]

Then

State equation:
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-10^8 & -16000
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
0
\end{bmatrix} g(t)
\]

Output equation:
\[
v = y = \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + [0]g
\]

**Example 2.** Consider the circuit shown below, where \( v_i(t) \) is the input voltage for \( t \geq 0 \), and there is zero energy stored in either inductor or capacitor at time \( t = 0 \).

The differential equation for the resistor voltage, \( v_0(t) \), is
\[
\frac{d^2 v_0}{dt^2} + \frac{1}{RC} \frac{dv_0}{dt} + \frac{1}{LC} v_0 = \frac{d^2 v_i}{dt^2} + \frac{1}{LC} v_i.
\]

Since the differential equation has derivatives of the input, the values of \( \beta \) are determined as
\[
\beta_0 = b_2 = 1, \quad \beta_1 = b_1 - a_1 \beta_0 = 0 - \frac{1}{RC} 1 = -\frac{1}{RC},
\]
\[
\beta_2 = b_0 - a_1 \beta_1 - a_0 \beta_0 = \frac{1}{LC} - \frac{1}{RC} \frac{1}{RC} - \frac{1}{LC} 1 = \frac{1}{(RC)^2}.
\]

The state variable matrices are then
\[
A = \begin{bmatrix}
0 & 1 \\
-1 & -\frac{1}{RC}
\end{bmatrix}, \quad B = \begin{bmatrix}
-\frac{1}{RC} \\
\frac{1}{(RC)^2}
\end{bmatrix}, \quad C = [1 \ 0], \quad D = [1].
\]

Matlab code is shown below for computing the circuit response to unit step and \( \cos(t)u(t) \) inputs. In both cases the initial conditions are \( x_1(0) = 0, \ x_2(0) = 0 \). Note that the circuit is a
notch filter, so the “dc gain” is unity, and the filter completely eliminates a 1 rad/sec sinusoid. This is evident in the numerical responses.

```matlab
function state_variables_example
% Example of state variables involving derivative of the input
% L = 1 H, Cap = 1 F, R = 1 Ohm.
L=1;Cap=1;R=1;
A=[0 1;-1/(L*Cap) -1/(R*Cap)];B=[-1/(R*Cap);1/(R*Cap)^2];C=[1 0];D=[1];
x0=[0;0];
t=[0:0.01:10];
vi=ones(1,length(t)); % unit step input
sys=ss(A,B,C,D); % Define the state space model
[y,x]=lsim(sys,vi,t,x0);
y1=y;
vi=cos(t); % w0 = 1 rad/sec is the notch frequency
[y,x]=lsim(sys,vi,t,x0);
figure(1)
plot(t,y1(:,1),t,y(:,1),'--')
xlabel('Time, t, sec')
ylabel('Voltage, v_0(t), V')
title('Resistor Voltage Output for Inputs v_i(t) = u(t) and v_i(t) = cos(t)')
legend('Step Response', 'Cosine Response')
```

Resistor Voltage Output for Inputs $v_i(t) = u(t)$ and $v_i(t) = \cos(t)$
An alternative approach to handling derivatives of the input is using the transfer function in Matlab. The same system simulation is shown below with unit step response and response to the input \( \sin(t) \).

```matlab
function state_variables_example
    % Example of MATLAB simulation if transfer function has zeros
    %   band-reject (notch) filter: \( H(s) = (s^2 + 1)/(s^2 + s + 1) \)
    sys=tf([1 0 1],[1 1 1]);
    t=[0:0.01:10];
    g=ones(1,length(t));
    figure(1)
    lsim(sys,g',t)
    title('Unit Step Response')
    g=sin(t);
    figure(2)
    lsim(sys,g',t)
    title('Response to Input \( \sin(t) \)')
end
```

![Unit Step Response](image1)

![Response to Input sin(t)](image2)