1. A periodic signal is given as
   \[ x(t) = 3 \cos(2\pi 8t) + 7 \sin(2\pi 16t). \]
   Find the period, fundamental frequency, and Fourier series.
   Sketch the magnitude spectrum.

   \[ T = \]
   \[ \omega_0 = \]
   \[ a_0 = \]
   \[ a_n = \]
   \[ b_n = \]
2. Find the Fourier series for the signal, x(t), shown below.

Plot the truncated Fourier series representation for N = 0, 1, 3, and 5.

Period, \( T = 0.004 \) sec.

Fundamental frequency = \( \omega_0 = \frac{2\pi}{T} \) rad/s (\( f_0 = \frac{1}{T} = 250 \) Hz).

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)
\]

\[
a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = \frac{1}{2}
\]

\[
b_n = 0
\]

\[
a_n = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \quad (a_n = 0 \text{ for } n \text{ even})
\]

\[
b_n = 0 \quad \text{(since } x(t) \text{ is an even function)}
\]

Truncated Fourier Series:

\[
x_N(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)
\]
Now, suppose that the phase of the Fourier series terms are incorrect. Specifically, suppose that for all $1 < n \leq N$ in the truncated Fourier series, the phases are off by exactly 180° in the magnitude/phase representation. This is shown below, compared to the correct truncated Fourier series, for $N = 3, 5, \text{ and } 15$. 

![Truncated Fourier Series](image)
Note that the phase errors cause the reconstructed waveforms to look quite different from the original square wave.
3. Conventional AM generated using a switching modulator. Problem: Use Fourier Series analysis to explain how the output signal in the circuit below, \( v_0(t) \), can be filtered to produce the “envelope” signal of the form \( A_c \cos(2\pi f_c t) + \mu m(t) \), where \( 0 \leq \mu \leq 1 \), and the message signal, \( m(t) \), is assumed scaled so that \( \max_t |m(t)| < 1 \).

![Switching Modulator Diagram]

Assume that the maximum value of \( |m(t)| \) is much less than \( A \). Then the signal across the resistor is well-modeled as

\[
v_0(t) = (A \cos(2\pi f_c t) + m(t)) g(t)
\]

where \( g(t) = \begin{cases} 1, & \text{if } \cos(2\pi f_c t) > 0; \\ 0, & \text{if } \cos(2\pi f_c t) < 0. \end{cases} \)

Hence, \( g(t) \) is a square wave, as in the previous problem, with period \( T = \frac{1}{f_c}, \omega_0 = \frac{2\pi}{T}, b_n = 0, \) and

\[
g(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{2}{n \pi} \sin \left( \frac{n \pi}{2} \right) \right) \cos(n \omega_0 t).
\]

As a specific example, suppose that the message signal is a cosine, say \( m(t) = \cos(2\pi f_m t) \), where \( f_m = 2,000 \) Hz, the carrier frequency is \( f_c = 50,000 \) Hz, and \( A = 100 \). In AM radio the carrier frequencies are at least 100 times larger than the highest frequency in the message (e.g., a carrier frequency of at least 540 kHz and a maximum message frequency of 5 kHz).
4. Find the Fourier series of the periodic signal, \( x(t) \), shown below.

\[ x(t) = x_e(t) + x_o(t) \]

**Solution:** Write \( x(t) = x_e(t) + x_o(t) \) and find the Fourier series of the even part and odd part of \( x(t) \), then combine.

\[ x_e(t) = \frac{x(t) + x(-t)}{2}, \quad x_o(t) = \frac{x(t) - x(-t)}{2} \]

The Fourier series coefficients are then computed as
\[ a_0 = 0 \] (for both \( x_e(t) \) and \( x_o(t) \)).

For the even part, \( b_n = 0 \) and \( a_n = \frac{8}{n\pi} \sin \left( \frac{n\pi}{2} \right) \).

For the odd part, \( a_n = 0 \) and \( b_n = \frac{-2}{n\pi} \left( 1 - \cos \left( \frac{n\pi}{2} \right) \right) \).

The complete Fourier series for \( x(t) \) is then \( a_0 = 0 \), \( a_n = \frac{8}{n\pi} \sin \left( \frac{n\pi}{2} \right) \), and \( b_n = \frac{-2}{n\pi} \left( 1 - \cos \left( \frac{n\pi}{2} \right) \right) \).
1. Use Matlab to plot the magnitude and phase spectra.

2. Determine how many terms are required in a truncated Fourier series to have 10% normalized MSE. To have 0.5% normalized MSE.

Magnitude: \( A_n = \sqrt{a_n^2 + b_n^2} \), Phase: \( \theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \),

```matlab
function fourier_series1(N1,N2)
% Computes the N1-term plot of the magnitude and phase spectra
% and the N2-term normalized MSE
a0=0;
n=[1:1:max(N1,N2)];
a=8*sin(0.5*pi*n)./(pi*n);
b=-2*(1-cos(0.5*pi*n))./(pi*n);
figure(1)
subplot(2,1,1)
stem([0 n(1:N1)],[a0 sqrt(a(1:N1).^2 + b(1:N1).^2)])
xlabel('Index, n')
ylabel('A_n')
title('Magnitude Spectrum')
subplot(2,1,2)
stem([0 n(1:N1)],[0 atan2(b(1:N1),a(1:N1))*180/pi])
xlabel('Index, n')
ylabel('\theta_n, Degrees')
title('Phase Spectrum')
normMSE=(4.5-(a0^2+0.5*(a(1:N2)*a(1:N2)' + b(1:N2)*b(1:N2))))/4.5;
Percent_Error = normMSE*100
```

\[
\text{MSE}_{\text{norm}} = \frac{P - \left[a_0^2 + \frac{1}{2} \sum_{n=1}^{N} a_n^2 + b_n^2 \right]}{P}
\]

10% \( \text{MSE}_{\text{norm}} \) \( \Rightarrow \) \( N = 5 \) terms

0.5% \( \text{MSE}_{\text{norm}} \) \( \Rightarrow \) \( N = 85 \) terms
3. Determine the fraction of the signal power in i) the first harmonic; in the first 3 harmonics.

\[
\text{Percent Power in } n^{th} \text{ harmonic} = \frac{0.5 \times (a_n^2 + b_n^2)}{P} \times 100\%
\]

Using \( P = 4.5 \),

\[
\text{Percent Power in 1st harmonic} = \frac{0.5 \times \left( \frac{8}{\pi} \right)^2 + \left( \frac{-2}{\pi} \right)^2}{4.5} \times 100 = 76.6\%
\]

\[
\text{Percent Power in first 3 harmonics} = \frac{0.5 \times (a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2)}{P} \times 100\%
\]

\[= 89.6\%
\]

4. Use Matlab to construct the truncated Fourier series,

\[x_N(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t)\]

5. Also, use Matlab to construct the truncated Fourier series from the magnitude/phase form of the Fourier series coefficients,

Magnitude: \( A_n = \sqrt{a_n^2 + b_n^2} \),

Phase: \( \theta_n = \tan^{-1} \left( \frac{b_n}{a_n} \right) \),

\[x_N(t) = a_0 + \sum_{n=1}^{N} A_n \cos(n \omega_0 t - \theta_n)\]

```matlab
function y=fs_example(N,T,time,original)
% Fourier series example
a0=0;
fs=48000; % Sampling rate, in samples per second.
n=[1:N];
a=(8/pi)*sin(n*pi/2)./n;
b=(-2/pi)*(1-cos(n*pi/2))./n;
t=[-T:1/fs:T+time];
len_N=floor(1+2*T*fs);
y=a0*ones(1,length(t));
for k=1:N
    y=y+a(k)*cos(k*2*pi*t/T)+b(k)*sin(k*2*pi*t/T);
end
```
y0 = y;
figure(1)
plot(t(1:len_N), y(1:len_N))
% Construct using magnitude/phase form of Fourier series
A = sqrt(a.^2 + b.^2);
theta = atan2(b, a);
y = a0*ones(1, length(t));
for k=1:N
  y = y + A(k)*cos(k*2*pi*t/T-theta(k));
end
6. Finally, randomize the phase, $\theta_n$, and construct the truncated Fourier series,

$$x_N(t) = a_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t - \tilde{\theta}_n),$$

where $\tilde{\theta}_n$ is $\theta_n$ plus a uniform random phase variable over $[-\pi, \pi]$.

function y=fs_example(N,T,time,original)
% Fourier series example
a0=0;
fs=48000; % Sampling rate, in samples per second.
n=[1:N];
a=(8/pi)*sin(n*pi/2)./n;
b=(-2/pi)*(1-cos(n*pi/2))./n;
t=[-T:1/fs:T+time];
len_N=floor(1+2*T*fs);
y=a0*ones(1,length(t));
for k=1:N
    y=y+a(k)*cos(k*2*pi*t/T)+b(k)*sin(k*2*pi*t/T);
end
y0=y;
figure(1)
plot(t(1:len_N),y(1:len_N))
% Construct using magnitude/phase form of Fourier series
A=sqrt(a.^2 + b.^2);
theta=atan2(b,a);
y=a0*ones(1,length(t));
for k=1:N
    y=y+A(k)*cos(k*2*pi*t/T-theta(k));
end
figure(2)
plot(t(1:len_N),y(1:len_N))
%Now randomize the phase
theta1=theta+2*pi*(rand(1,length(theta))-0.5);
figure(3)
y=a0*ones(1,length(t));
for k=1:N
    y=y+A(k)*cos(k*2*pi*t/T-theta1(k));
end
plot(t(1:len_N),y(1:len_N))
if(original==0)
y=y0;
end

>> y=fs_example(19,0.004,2,0); % Output the truncated Fourier series waveform
>> y1=fs_example(19,0.004,2,1); % Output the Fourier series with random phase

% Play out the sound (the sampling rate is 48,000 samples per second)

>> sound(y/50,48000)
>> sound(y1/50,48000)

Note: Each time the program is run, a different random set of phases is generated, and a different reconstructed waveform generated.
Each time the program is run, a different reconstructed random waveform generated.
Speech Coding Example: Segment of speech file (top) and segment of improved multi-band excitation (IMBE) coded speech (bottom). The original speech file is 12-bit PCM with a sampling rate of 8,000 samples/sec. “Toll-quality” μ-law PCM uses 8 bits/sample, and a transmission rate of 64 kb/s. The IMBE speech is encoded at rate 4,800 bits/s.