Preliminary Report Solutions Outline

A circuit with input voltage, \( v_i(t) \), and output voltage, \( v_o(t) \), is shown below.

![Circuit Diagram](image)

Figure 1. Circuit with input voltage, \( v_i(t) \), and output voltage, \( v_o(t) \).

**Part 1.**

1. Assume all-zero initial conditions and use s-domain techniques to determine

\[
H(s) = \frac{v_o(s)}{v_i(s)}
\]

Solution: Use KVL around the two loops, with \( I_1(s) \) and \( I_2(s) \) the loop currents.

\[
V_i(s) = I_1(Z_1 + Z_2) - I_2(Z_2) \\
0 = -I_1Z_2 + I_2(Z_2 + Z_3 + Z_4)
\]

Writing as a matrix equation and using Cramer’s rule yields

\[
H(s) = \frac{V_0(s)}{V_i(s)} = \frac{Z_2Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}
\]

Substituting for the component values yields the transfer function

\[
H(s) = \frac{V_0(s)}{V_i(s)} = \frac{R_2}{s^3 + \left(\frac{1}{C_1R_1} + \frac{1}{C_2R_2}\right)s^2 + \left(\frac{1}{C_1C_2R_1R_2} + \frac{1}{LC_1} + \frac{1}{LC_2}\right)s + \frac{(R_1 + R_2)}{LC_1C_2R_1R_2}}
\]

2. Using Kirchhoff’s voltage and currents laws, derive a 3rd order differential equation for \( v_o(t) = v_{c_2}(t) \) in the circuit above. Assume that there is no energy stored in the capacitors or inductor at time \( t = 0^- \). From the differential equation, find a state variable representation, and specify the state variable matrices A, B, C, D, to generate the “outputs,” voltages \( v_o(t) \) and \( v_L(t) \).

Derivation of differential equation attached.

3. Let the circuit elements have parameter values \( L = (0.15 + 0.4\alpha)H \), \( C_1 = 1 \mu F \), \( C_2 = 250 \, nF \), \( R_1 = 50 \, \Omega \), and \( R_2 = 1,000 + 4000 \alpha \, \Omega \), where \( \alpha = 0 \)xyz, with xyz the
The final three digits of your student ID number. The value of $0.\text{xyz}$ satisfies $0 \leq 0.\text{xyz} \leq 1$, so the value of $L$ lies in the range $[0.15, 0.55] H$, and the value of $R_2$ lies in the range $[1, 5] k\Omega$. Use Matlab to determine the response of the circuit ($v_o(t)$ and $v_L(t)$) over a suitable time interval (roughly $[0, 10]$ msec) to input $v_i(t) = u(t) V$, where $u(t)$ is the unit-step signal. Determine the "dc" gain of the filter, $g$, and accurately plot the scaled unit step response, $v_o(t)/g$. Determine (from the scaled numerical response) the 100% rise time, percent overshoot, and 2% settling time. (Note that the default rise time value that some Matlab functions provide is the 10% to 90% rise time, so make sure your numerical result matches your plot and corresponds to the 100% rise time.) Verify that $v_o(t)$ and $v_L(t)$ have the correct values at $t = 0^+$ and approach the correct steady-state values as $t \to \infty$.

Plot of unit step response for $\alpha = 0, R_1 = 1,000 \, \Omega, L = 0.15 \, H$, and $\alpha = 1, R_1 = 5,000 \, \Omega, L = 0.55 \, H$. The respective rise time, settling time, and percent overshoot are $t_r = 0.46 \, ms, t_s = 1.63 \, ms, M_p = 23.9\%$ for $R_1 = 1,000 \, \Omega, L = 0.15 \, H$, and $t_r = 0.698 \, ms, t_s = 8.5 \, ms, M_p = 58.7\%$ for $R_1 = 5,000 \, \Omega, L = 0.55 \, H$. 
function projS2018(R1,R2,L,Cap1,Cap2,alpha)
% 3rd-order passive filter with parallel RC
% combination at output
% projS2018(50,1000,.15,1e-6,250e-9)
b0=R2/(R2*R1*L*Cap1*Cap2);
a2=1/(R1*Cap1) + 1/(R2*Cap2);
a1=1/(R1*R2*Cap1*Cap2) + 1/(L*Cap1) + 1/(L*Cap2);
a0=(R1+R2)/(R2*R1*L*Cap1*Cap2);
gain=b0/a0;
t=[0:1e-7:7.10e-3];
vi=ones(1,length(t));
% state variables approach
A=[0 1 0;0 0 1;-a0 -a1 -a2];B=[0;0;b0];
C=[1 0 0;L/R2 L*Cap2;1 L/R2 L*Cap2];D=[0;0;0];
[y1,x1]=lsim(A,B,C,D,vi,t,[0;0;0]);
figure(1)
plot(t,y1(:,1)/gain,t,y1(:,2),'--')
title('Unit Step Response, v_0(t)/g, v_L(t), R_2 = 5000')
xlabel('Time, t, sec.')
ylabel('Voltage, v_0(t)/g, v_L(t)')
legend('v_0(t)/g', 'v_L(t)')
end
Derivation of differential equation for $V_o(t) = Z \Sigma V_i(t)$.

KVL: loop 1

$\sum V_i - \frac{V_1}{R_1} + V_c = 0$

$\sum V_i = 0$

loop 2

$V_1 = V_2 + V_c$

$L \frac{d^2 V_2}{dt^2} + \frac{V_2}{R_2} + C \frac{d V_2}{dt} = V_2$

KCL:

$i_2 = \frac{V_2}{R_2} + \frac{C \frac{d V_2}{dt}}{R_2}$

Simplifying:

$V = R_1 C_2 \frac{d^3 V_2}{dt^3} + \frac{R_2}{R_1} \frac{d^2 V_2}{dt^2} + \frac{C_1}{R_1} \frac{d V_2}{dt} + \frac{C_2}{R_2} V_2$

Finally:

$\frac{d^3 V_2}{dt^3} + \left( \frac{1}{R_1} + \frac{C_1}{C_2} \right) \frac{d^2 V_2}{dt^2} + \left( \frac{1}{C_2} + \frac{1}{R_2} \right) \frac{d V_2}{dt} + \frac{1}{R_1 C_1 R_2} V_2 = \frac{1}{R_1 C_1 R_2} V_2$

This yields the same transfer function as in part 1, $H(s) = \frac{V_o(s)}{V_i(s)}$. 
