Final Exam

EE 341

Fall 2007

Closed book, two hour exam. Three (double-sided) pages of notes are allowed. Some useful formulae are appended. There are 11 problems for a total of 210 possible points.

1. A signal, \( x(t) \), has the spectrum shown below. An amplitude modulation signal is formed as \( y(t) = A[1 + \mu x(t)] \cos(2\pi f_c t) \), where \( f_c = 980 \text{ kHz} \), \( 0 < \mu < 1 \), and \( \max_t |x(t)| < 1 \).

\[
\begin{array}{c}
\text{X(w)} \\
\end{array}
\]

a) (5 points) What is the bandwidth of the AM signal?

b) (10 points) Find \( Y(\omega) \) in terms of \( X(\omega) \) and accurately sketch \( Y(\omega) \), labeling relevant amplitudes and frequencies.
2. (10 points) A linear, time-invariant system with impulse response \( h(t) \) has the frequency response \( H(\omega) = A(\omega)e^{-j\omega t_0} \), where \( A(\omega) \) is real-valued, as shown below, \( t_0 = 0.0001 \text{ sec} \), and the phase is also shown below.

Find the system response to the input \( x(t) = \cos(2\pi 250t) \).
3. A signal $x(t)$ is applied as input to a sampler, followed by an ideal lowpass filter, as shown below.

![Diagram](image)

The reconstructed signal is $y(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\frac{t-nT}{T}\right)$.

a) (5 points) If $X(\omega)$ is as shown below, according to the sampling theorem what is the minimum sampling rate, $1/T$, to avoid aliasing distortion?

![Spectrum](image)

b) (10 points) Let $x(t) = 2\sin(2\pi f_1 t) + 3\cos(2\pi f_2 t)$, with $f_1 = 100$ Hz, $f_2 = 250$ Hz, and sampling rate $1/T = 400$ samples/sec. **Sketch** the spectrum at the output of the sampler and **find** $y(t)$. Simplify the expression for $y(t)$ as much as possible.
4. (15 points) Design an analog notch filter to remove the frequency 1000 Hz. Specifically, design a two-pole, two-zero notch filter. Specify the filter transfer function, $H(s)$, identify the pole and zero locations and sketch their location in the $s$-plane, and sketch the frequency response for your filter, labeling relevant amplitudes and frequencies.
5. (30 points) A linear, time-invariant system has the impulse response given below (two cases - one continuous-time, and one discrete-time). Determine, for each of the impulse responses, whether or not the system is i) memoryless, ii) causal, and iii) BIBO stable. **Valid justification must be given to receive credit.**

a) (15 points) Discrete-time

\[ h[n] = \begin{cases} 
  n^2, & 0 \leq n \leq 10 \\
  0, & \text{otherwise}
\end{cases} \]

b) (15 points) Continuous-time

\[ h(t) = \begin{cases} 
  \frac{2}{1+|t|}, & \text{for } -1 \leq t \\
  0, & \text{otherwise}
\end{cases} \]
6. a. (15 points) **Find** the discrete convolution of $h(n)$ and $x(n)$ given below.

$$h(n) = (\cdots, 0, -1, 1, 1, -1, 0, 0, \cdots), \quad x(n) = (\cdots, 0, 0, 1, 2, 3, 4, 3, 2, 1, 0, 0, \cdots)$$

b. (15 points) **Find** $y(t) = h(t) * x(t)$ for the signals given below. **Sketch** $y(t)$.

$$h(t) = 2e^{-(t-1)}u(t - 1), \quad x(t) = u(t) - u(t - 5)$$
7. (20 points) A discrete-time, linear, time-invariant system with input \( x(n) \) and output \( y(n) \) has the realization shown below.

![System Diagram](image)

a) (5 points) Find the difference equation for the system.

b) (5 points) Find the system impulse response, \( h(n) \).

c) (5 points) Find the system output, \( y(n) \), to the unit-step input,

\[
x(n) = u(n) = \begin{cases} 
1, & \text{for } n \geq 0; \\
0, & \text{for } n < 0. 
\end{cases}
\]

d) (5 points) Find the system frequency response, \( H(e^{j\Omega}) \), which is the discrete-time Fourier Transform of \( h(n) \). Sketch \( |H(e^{j\Omega})| \) over one period of the (normalized) frequency range \([-\pi, \pi]\).
8. (20 points) A periodic signal, \( x(t) \), is shown below. (Note that the time axis is labeled in units of milliseconds.)

\[
\begin{array}{c|c}
-2 & 1 \\
-1 & 1 \\
1 & 2 \\
2 & 3 \\
3 & 4 \\
\hline
\text{t, ms} & x(t)
\end{array}
\]

a) Determine the exponential Fourier series representation

\[
x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T}.
\]

Specifically, determine
i) (2 points) The signal period, \( T \).
ii) (3 points) The Fourier series coefficient \( c_0 \).
iii) (7 points) For \( n \neq 0 \), the Fourier series coefficients \( c_n \).

b) (8 points) The signal \( x(t) \) is passed through a linear, time-invariant filter with frequency response

\[
H(\omega) = \begin{cases} 
e^{-j\omega t_0} , & \text{for } 1000\pi < |\omega| < 1500\pi; \\
0 , & \text{otherwise}, 
\end{cases}
\]

where \( t_0 = 0.00025 \text{ second} \).

Find the filter output, \( y(t) \). Simplify as much as possible.
9. (20 points) A discrete memoryless source, $X$, has an alphabet of size 8, with $P(X = 1) = 0.35$, $P(X = 2) = 0.175$, $P(X = 3) = 0.175$, $P(X = 4) = 0.08$, $P(X = 5) = 0.08$, and $P(X = 6) = 0.08$, $P(X = 7) = 0.04$, and $P(X = 8) = 0.02$,

a. (5 points) Compute the entropy, $H(X)$, in bits.

b. (15 points) Design a Huffman code for this source. Specify the codeword for each source letter, and compute the average number of codeword bits per source letter, $\bar{\ell}$. Compare the average codeword length to the entropy.
10. (15 points) A baseband (PAM) digital signaling system is to be designed to transmit 4,800 bits/sec over a baseband channel of bandwidth 1,100 Hz (i.e., the frequency band [0, 1100] Hz is available). Assume a raised cosine pulse with spectral rolloff parameter $\alpha$ is to be used, and determine a suitable design for the system by specifying:
   i) the number of bits per symbol, $R$;
   ii) the symbol rate, $1/T$;
   iii) the value of $\alpha$ for the raised cosine pulse;
   iv) the required bandwidth for your design.
11. (20 points) Design a Butterworth filter to satisfy the following frequency response specifications:
   i) the gain is no less than -1 dB for $|f| \leq 2,000$ Hz.
   ii) the gain is no larger than -25 dB for $|f| > 5,000$ Hz.

For your design, determine $N$, the required number of poles in the Butterworth filter transfer function, and $\omega_c$, the 3 dB cutoff frequency.
Useful Formulae

Define the “boxcar” time and frequency pulses as

\[ p_T(t) = \begin{cases} 1, & \text{for } |t| \leq T; \\ 0, & \text{otherwise}. \end{cases} \]

\[ P_W(\omega) = \begin{cases} 1, & \text{for } |\omega| \leq W; \\ 0, & \text{otherwise}. \end{cases} \]

Define the sinc pulse as

\[ \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}. \]

Various Fourier Transform Relations

\[ \mathcal{F}\{p_T(t)\} = 2T \cdot \text{sinc}\left(\frac{\omega T}{\pi}\right) \]

\[ \mathcal{F}\{\cos(\omega_c t)\} = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \]

\[ \mathcal{F}\{x(t - t_0)\} = X(\omega) e^{-j\omega t_0} \]

\[ \mathcal{F}\{e^{-at} u(t)\} = \frac{1}{a + j\omega} \]

\[ \mathcal{F}\{x(t) y(t)\} = \frac{1}{2\pi} X(\omega) * Y(\omega) \]

\[ \mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2 + \omega^2} \]

\[ \mathcal{F}\left\{ \frac{W}{\pi} \text{sinc}\left(\frac{W t}{\pi}\right) \right\} = P_W(\omega) \quad \mathcal{F}\{\text{sinc}(\frac{t}{T})\} = P_{\frac{T}{T}}(\omega) \]

\[ |H(\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2N}} \]

\[ h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k) x(n - k) \]

\[ H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h(n) e^{-j\Omega k} \]

\[ f = ma \quad a = \pi r^2 \quad e = mc^2 \]

\[ X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \]

Raised Cosine Pulse Bandwidth: \( BW = \frac{1 + a}{2T} \)