Horner's Rule Loop Invariant
CptS 355

Fall 2005

Here's another example of how a loop invariant works. This example looks at an iterative version of the Horner's rule algorithm. We assume that the polynomial to be evaluated is $s = \sum_{j=0}^{n} a_j x^j$. The algorithm is

```
{P}
i = n
s = 0
{I}
while (i>=0) do
   {I ∧ i>=0}
   s = s*x + a_i
   i = i-1
end
{I ∧ i<0}
{s = Σ_{j=0}^{n} a_j x^j}
```

There are three major steps to showing that the program does what we want:

1. Find a candidate for the loop invariant and show that it actually is an invariant.
2. Show that the loop invariant together with the termination condition of the loop imply that the desired result has been achieved.
3. Figure out a precondition for the program that ensures the loop invariant is true when execution reaches the beginning of the loop.

Step 1. Find a candidate loop invariant and show it actually is an invariant

So, we need to find a predicate for the invariant, {I}. Let's build a table of values of values for i and s to see if we can discover a relationship that can be captured in an invariant:

i	s
n	0
n-1	a_n
n-2	$a_{n-1} + a_n x$
n-3	$a_{n-2} + a_{n-1}x + a_n x^2$

For the invariant let's try $s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i \geq -1$. Is it an invariant? To find out we have to see if the invariant *and* the loop test imply wp(loop body, invariant). That is, does

$$s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i \geq -1 \wedge i \geq 0$$

imply

$$wp(s = s * x + a_i;\ i = i - 1,\ \{s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i \geq -1\})$$

So what is this weakest precondition? Let's figure it out:

$$wp(s = s * x + a_i;\ i = i - 1,\ \{s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i \geq -1\}) \equiv$$

$$wp(s = s * x + a_i,\ \{s = \sum_{j=i}^{n} a_j x^{j-i} \wedge i \geq 0\}) \equiv$$

$$s * x + a_i = \sum_{j=i}^{n} a_j x^{j-i} \wedge i \geq 0 \equiv$$

$$s * x = \sum_{j=i+1}^{n} a_j x^{j-i} + a_i x^{i-i} - a_i \wedge i \geq 0 \equiv$$

$$s = \frac{\sum_{j=i+1}^{n} a_j x^{j-i}}{x} \wedge i \geq 0 \equiv$$

$$s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i \geq 0$$

so $s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i \geq -1$ is indeed an invariant for this loop.

Step 2: Show that the loop invariant and loop termination imply the desired result

When the loop terminates we know that the invariant *and not* the loop test are true. What does this tell us about the state at the end of the program?

$$s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i \geq -1 \wedge i < 0 \equiv$$

$$s = \sum_{j=i+1}^{n} a_j x^{j-i-1} \wedge i = -1 \equiv$$

$$s = \sum_{j=0}^{n} a_j x^{j}$$

which is the desired result.

Step 3: Determine a precondition for the program that ensures the invariant holds at the beginning of the loop

If the invariant is to be true when the loop test is first executed what needs to be true when the program starts? That is, what is the precondition {P} in the annotated algorithm? We need to figure out

$$wp(i = n;\ s = 0,\ \{s = \sum_{j=i+1}^{n} a_j x^{j-i-1}\}) \equiv$$

$$wp(i = n, \{0 = \sum_{j=i+1}^{n} a_j x^{j-i-1}\}) \equiv$$

$$0 = \sum_{j=n+1}^{n} a_j x^{j-n-1}$$

which is *true* in all states, so we conclude this program computes the value of the polynomial regardless of the initial state.