Applications

- Depth-first search
- Biconnectivity
- Euler circuits
- Strongly-connected components
Depth-First Search

- Recursively visit every vertex in the graph
- Considers every edge in the graph
  - Assumes undirected edge \((u,v)\) is in \(u\)'s and \(v\)'s adjacency list
- Visited flag prevents infinite loops
- Running time \(O(|V|+|E|)\)

```plaintext
DFS () ;; graph G=(V,E)
    foreach v in V
        if (! v.visited)
            then Visit (v)

Visit (vertex v)
    v.visited = true
    foreach w adjacent to v
        if (! w.visited)
            then Visit (w)
```
DFS Applications

- Undirected graph
  - Test if graph is connected
    - Run DFS from any vertex and then check if any vertices not visited
  - Depth-first spanning tree
    - Add edge \((v,w)\) to spanning tree if \(w\) not yet visited (minimum spanning tree?)
    - If graph not connected, then depth-first spanning forest
Remembering the DFS traversal order is important for many applications.

Let the edges \((v,w)\) added to the DF spanning tree be directed.

Add a directed back edge (dashed) if:
- \(w\) is already visited when considering edge \((v,w)\), and
- \(v\) is already visited when considering reverse edge \((w,v)\).
Biconnectivity

- A connected, undirected graph is **biconnected** if the graph is still connected after removing any one vertex
  - I.e., when a “node” fails, there is always an alternative route

- If a graph is not biconnected, the disconnecting vertices are called **articulation points**
  - Critical points of interest in many applications
DFS Applications: Finding Articulation Points

- From any vertex v, perform DFS and number vertices as they are visited
  - Num(v) is the visit number
- Let Low(v) = lowest-numbered vertex reachable from v using 0 or more spanning tree edges and then at most one back edge
  - Low(v) = minimum of
    - Num(v)
    - Lowest Num(w) among all back edges (v,w)
    - Lowest Low(w) among all tree edges (v,w)
- Can compute Num(v) and Low(v) in $O(|E| + |V|)$ time
DFS Applications: Finding Articulation Points (Example)

Original Graph

Depth-first tree starting at A with Num/Low values:
DFS Applications: Finding Articulation Points

- **Root** is an articulation point iff it has more than one child.
- Any other vertex \( v \) is an articulation point iff \( v \) has some child \( w \) such that \( \text{Low}(w) \geq \text{Num}(v) \).
  - I.e., is there a child \( w \) of \( v \) that cannot reach a vertex visited before \( v \)?
  - If yes, then removing \( v \) will disconnect \( w \) (and \( v \) is an articulation point).
DFS Applications: Finding Articulation Points (Example)

Original Graph

Depth-first tree starting at C with Num/Low values:
DFS Applications: Finding Articulation Points

- High-level algorithm
  - Perform pre-order traversal to compute Num
  - Perform post-order traversal to compute Low
  - Perform another post-order traversal to detect articulation points

- Last two post-order traversals can be combined

- In fact, all three traversals can be combined in one recursive algorithm
/**
 * Assign num and compute parents.
 */

void Graph::assignNum( Vertex v )
{
    v.num = counter++;
    v.visited = true;
    for each Vertex w adjacent to v
        if( !w.visited )
            {
                w.parent = v;
                assignNum( w );
            }
}
/**
 * Assign low; also check for articulation points.
 */
void Graph::assignLow( Vertex v )
{
    v.low = v.num; // Rule 1
    for each Vertex w adjacent to v
    {
        if( w.num > v.num ) // Forward edge
        {
            assignLow( w );
            if( w.low >= v.num )
            {
                cout << v << " is an articulation point" << endl;
                v.low = min( v.low, w.low ); // Rule 3
            }
        }
        else
        {
            if( v.parent != w ) // Back edge
                v.low = min( v.low, w.num ); // Rule 2
        }
    }
}
void Graph::findArt( Vertex v )
{
    v.visited = true;
    v.low = v.num = counter++;  // Rule 1
    for each Vertex w adjacent to v
    {
        if( !w.visited )  // Forward edge
        {
            w.parent = v;
            findArt( w );
            if( w.low >= v.num )
                cout << v << " is an articulation point" << endl;
            v.low = min( v.low, w.low );  // Rule 3
        }
        else
        {
            if( v.parent != w )  // Back edge
                v.low = min( v.low, w.num );  // Rule 2
        }
    }
}
Euler Circuits

- Puzzle challenge
  - Can you draw a figure using a pen, drawing each line exactly once, without lifting the pen from the paper?
  - And, can you finish where you started?
Euler Circuits

- Solved by Leonhard Euler in 1736 using a graph approach (DFS)
  - Also called an “Euler path” or “Euler tour”
  - Marked the beginning of graph theory
Euler Circuit Problem

- Assign a vertex to each intersection in the drawing
- Add an undirected edge for each line segment in the drawing
- Find a path in the graph that traverses each edge exactly once, and stops where it started
Euler Circuit Problem

- Necessary and sufficient conditions
  - Graph must be connected
  - Each vertex must have an even degree
- Graph with two odd-degree vertices can have an Euler tour (not circuit)
- Any other graph has no Euler tour or circuit

![Graphs illustrating Euler circuit problem](image-url)
Euler Circuit Problem

- Algorithm
  - Perform DFS from some vertex \( v \) until you return to \( v \) along path \( p \)
  - If some part of graph not included, perform DFS from first vertex \( v' \) on \( p \) that has an un-traversed edge (path \( p' \))
  - Splice \( p' \) into \( p \)
  - Continue until all edges traversed
Euler Circuit Example

Start at vertex 5.
Suppose DFS visits 5, 4, 10, 5.
Graph remaining after 5, 4, 10, 5:

Start at vertex 4.
Suppose DFS visits 4, 1, 3, 7, 4, 11, 10, 7, 9, 3, 4.
Splicing into previous path: 5, 4, 1, 3, 7, 4, 11, 10, 7, 9, 3, 4, 10, 5.
Graph remaining after 5, 4, 1, 3, 7, 4, 11, 10, 7, 9, 3, 4, 10, 5:

Start at vertex 3.
Suppose DFS visits 3, 2, 8, 9, 6, 3.
Splicing into previous path: 5, 4, 1, 3, 2, 8, 9, 6, 3, 7, 4, 11, 10, 7, 9, 3, 4, 10, 5.
Euler Circuit Example (cont.)

Graph remaining after 5, 4, 1, 3, 2, 8, 9, 6, 3, 7, 4, 11, 10, 7, 9, 3, 4, 10, 5:

Start at vertex 9.
Suppose DFS visits 9, 12, 10, 9.
Splicing into previous path: 5, 4, 1, 3, 2, 8, 9, 12, 10, 9, 6, 3, 7, 4, 11, 10, 7, 9, 3, 4, 10, 5.
No more un-traversed edges, so above path is an Euler circuit.
Euler Circuit Algorithm

- Implementation details
  - Maintain circuit as a linked list to support O(1) splicing
  - Maintain index on adjacency lists to avoid repeated searches for un-traversed edges

- Analysis
  - Each edge considered only once
  - Running time is $O(|E| + |V|)$
DFS on Directed Graphs

- Same algorithm
- Graph may be connected, but not strongly connected
- Still want the DF spanning forest to retain information about the search

```
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    foreach v in V
        if (! v.visited)
            then Visit (v)

Visit (vertex v)
    v.visited = true
    foreach w adjacent to v
        if (! w.visited)
            then Visit (w)
```

```
A
  ^
  |
  v
B

D
  |
  v
C

E
```
DF Spanning Forest

- Three types of edges in DF spanning forest
  - **Back edges** linking a vertex to an ancestor
  - **Forward edges** linking a vertex to a descendant
  - **Cross edges** linking two unrelated vertices

Graph:

![Graph](image1)

DF Spanning Forest:

![Forest](image2)
DF Spanning Forest

(Note: DF Spanning Forests usually drawn with children and new trees added from left to right.)
DFS on Directed Graphs

- Applications
  - Test if directed graph is acyclic
    - Has no back edges
  - Topological sort
    - Reverse post-order traversal of DF spanning forest
Strongly-Connected Components

- A graph is strongly connected if every vertex can be reached from every other vertex.
- A strongly-connected component of a graph is a subgraph that is strongly connected.
- Would like to detect if a graph is strongly connected.
- Would like to identify strongly-connected components of a graph.
- Can be used to identify weaknesses in a network.
- General approach: Perform two DFSs.
Strongly-Connected Components

Algorithm

- Perform DFS on graph $G$
  - Number vertices according to a post-order traversal of the DF spanning forest
- Construct graph $G_r$ by reversing all edges in $G$
- Perform DFS on $G_r$
  - Always start a new DFS (initial call to Visit) at the highest-numbered vertex
- Each tree in resulting DF spanning forest is a strongly-connected component
Strongly-Connected Components

Graph G

Graph $G_r$

DF Spanning Forest of $G_r$

Strongly-connected components:

\{G\}, \{H,I,J\}, \{B,A,C,F\}, \{D\}, \{E\}
Strongly-Connected Components: Analysis

- Correctness
  - If $v$ and $w$ are in a strongly-connected component
  - Then there is a path from $v$ to $w$ and a path from $w$ to $v$
  - Therefore, there will also be a path between $v$ and $w$ in $G$ and $G_r$

- Running time
  - Two executions of DFS
  - $O(|E| + |V|)$
Summary

- Graphs one of the most important data structures
- Studied for centuries
- Numerous applications
- Some of the hardest problems to solve are graph problems
  - E.g., Hamiltonian (simple) cycle, Clique