Bayesian networks

Chapter 14.1–3
Outline

◇ Syntax
◇ Semantics
◇ Parameterized distributions
Bayesian networks

A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions

Syntax:
- a set of nodes, one per variable
- a directed, acyclic graph (link $\approx$ “directly influences”)
- a conditional distribution for each node given its parents:
  \[ P(X_i|Parents(X_i)) \]

In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over $X_i$ for each combination of parent values
Example

Topology of network encodes conditional independence assertions:

![Graph with nodes: Weather, Cavity, Toothache, Catch. Arrows indicate dependencies.]

*Weather* is independent of the other variables

*Toothache* and *Catch* are conditionally independent given *Cavity*
Example

I’m at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn’t call. Sometimes it’s set off by minor earthquakes. Is there a burglar?

Variables: Burglar, Earthquake, Alarm, JohnCalls, MaryCalls

Network topology reflects “causal” knowledge:
- A burglar can set the alarm off
- An earthquake can set the alarm off
- The alarm can cause Mary to call
- The alarm can cause John to call
Example contd.

| B | E | P(A|B,E) |
|---|---|---------|
| T | T | .95     |
| T | F | .94     |
| F | T | .29     |
| F | F | .001    |

| A | P(J|A) |
|---|------|
| T | .90  |
| F | .05  |

| A | P(M|A) |
|---|------|
| T | .70  |
| F | .01  |
Compactness

A CPT for Boolean $X_i$ with $k$ Boolean parents has $2^k$ rows for the combinations of parent values.

Each row requires one number $p$ for $X_i = true$ (the number for $X_i = false$ is just $1 - p$).

If each variable has no more than $k$ parents, the complete network requires $O(n \cdot 2^k)$ numbers.

I.e., grows linearly with $n$, vs. $O(2^n)$ for the full joint distribution.

For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)
Global semantics defines the full joint distribution as the product of the local conditional distributions:

\[ P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | \text{parents}(X_i)) \]

e.g., \[ P(j \land m \land a \land \neg b \land \neg e) \]

=  

\[
\begin{tikzpicture}[thick, scale=0.6]
  \node (A) at (0,0) {A};
  \node (B) at (-2,-2) {B};
  \node (E) at (2,-2) {E};
  \node (J) at (-2,2) {J};
  \node (M) at (2,2) {M};

  \draw (A) edge (B)
  \draw (A) edge (E)
  \draw (A) edge (J)
  \draw (A) edge (M);
\end{tikzpicture}
\]
“Global” semantics defines the full joint distribution as the product of the local conditional distributions:

\[ P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | \text{parents}(X_i)) \]

e.g., \( P(j \land m \land a \land \neg b \land \neg e) \)

\[ = P(j | a) P(m | a) P(a | \neg b, \neg e) P(\neg b) P(\neg e) \]
\[ = 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \]
\[ \approx 0.00063 \]
Local semantics: each node is conditionally independent of its nondescendants given its parents

Theorem: Local semantics $\iff$ global semantics
Markov blanket

Each node is conditionally independent of all others given its
Markov blanket: parents + children + children’s parents
Constructing Bayesian networks

Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics

1. Choose an ordering of variables $X_1, \ldots, X_n$
2. For $i = 1$ to $n$
   - add $X_i$ to the network
   - select parents from $X_1, \ldots, X_{i-1}$ such that
     \[
     P(X_i|\text{Parents}(X_i)) = P(X_i|X_1, \ldots, X_{i-1})
     \]

This choice of parents guarantees the global semantics:

\[
P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \quad \text{(chain rule)}
\]
\[
= \prod_{i=1}^{n} P(X_i|\text{Parents}(X_i)) \quad \text{(by construction)}
\]
Example

Suppose we choose the ordering $M, J, A, B, E$

$P(J|M) = P(J)$?
Example

Suppose we choose the ordering $M, J, A, B, E$

\begin{align*}
  P(J|M) &= P(J) \quad \text{No} \\
  P(A|J, M) &= P(A|J) \quad P(A|J, M) = P(A) \quad ?
\end{align*}
Suppose we choose the ordering $M, J, A, B, E$

$P(J|M) = P(J)$? No
$P(B|A, J, M) = P(B|A)$?
$P(B|A, J, M) = P(B)$?
Example

Suppose we choose the ordering $M, J, A, B, E$

$P(J|M) = P(J)\text{? No}$

$P(A|J, M) = P(A|J)? P(A|J, M) = P(A)\text{? No}$

$P(B|A, J, M) = P(B|A)? \text{Yes}$

$P(B|A, J, M) = P(B)\text{? No}$

$P(E|B, A, J, M) = P(E|A)?$

$P(E|B, A, J, M) = P(E|A, B)?$
Example

Suppose we choose the ordering $M, J, A, B, E$

$P(J|M) = P(J)$? No
$P(B|A, J, M) = P(B|A)$? Yes
$P(B|A, J, M) = P(B)$? No
$P(E|B, A, J, M) = P(E|A)$? No
$P(E|B, A, J, M) = P(E|A, B)$? Yes
Deciding conditional independence is hard in noncausal directions

(Causal models and conditional independence seem hardwired for humans!)

Assessing conditional probabilities is hard in noncausal directions

Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed
Example: Car diagnosis

Initial evidence: car won’t start
Testable variables (green), “broken, so fix it” variables (orange)
Hidden variables (gray) ensure sparse structure, reduce parameters
Example: Car insurance
Compact conditional distributions

CPT grows exponentially with number of parents
CPT becomes infinite with continuous-valued parent or child

Solution: canonical distributions that are defined compactly

Deterministic nodes are the simplest case:
\[ X = f(Parents(X)) \text{ for some function } f \]

E.g., Boolean functions
\[ \text{North American} \Leftrightarrow \text{Canadian} \lor \text{US} \lor \text{Mexican} \]

E.g., numerical relationships among continuous variables
\[ \frac{\partial \text{Level}}{\partial t} = \text{inflow} + \text{precipitation} - \text{outflow} - \text{evaporation} \]
Noisy-OR distributions model multiple noninteracting causes

1) Parents $U_1 \ldots U_k$ include all causes (can add leak node)
2) Independent failure probability $q_i$ for each cause alone

$$P(X|U_1 \ldots U_j, \neg U_{j+1} \ldots \neg U_k) = 1 - \prod_{i=1}^j q_i$$

<table>
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<th></th>
<th>Cold</th>
<th>Flu</th>
<th>Malaria</th>
<th>$P(\text{Fever})$</th>
<th>$P(\neg\text{Fever})$</th>
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<td>0.02 = 0.2 $\times$ 0.1</td>
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<td></td>
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<tr>
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<td>F</td>
<td>T</td>
<td></td>
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<tr>
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<td>F</td>
<td></td>
<td>0.88</td>
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<tr>
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<td>T</td>
<td>T</td>
<td></td>
<td>0.988</td>
<td>0.012 = 0.6 $\times$ 0.2 $\times$ 0.1</td>
</tr>
</tbody>
</table>

Number of parameters **linear** in number of parents
Hybrid (discrete+continuous) networks

Discrete (Subsidy? and Buys?); continuous (Harvest and Cost)

Option 1: discretization—possibly large errors, large CPTs
Option 2: finitely parameterized canonical families

1) Continuous variable, discrete+continuous parents (e.g., Cost)
2) Discrete variable, continuous parents (e.g., Buys?)
Continuous child variables

Need one conditional density function for child variable given continuous parents, for each possible assignment to discrete parents

Most common is the linear Gaussian model, e.g.,:

\[
P(Cost = c | Harvest = h, Subsidy? = true) = N(a_th + b_t, \sigma_t)(c) = \frac{1}{\sigma_t \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{c - (a_th + b_t)}{\sigma_t} \right)^2 \right)
\]

Mean Cost varies linearly with Harvest, variance is fixed

Linear variation is unreasonable over the full range
  but works OK if the likely range of Harvest is narrow
Continuous child variables

All-continuous network with LG distributions
  ⇒ full joint distribution is a multivariate Gaussian

Discrete+continuous LG network is a conditional Gaussian network i.e., a multivariate Gaussian over all continuous variables for each combination of discrete variable values
Discrete variable w/ continuous parents

Probability of $Buys?$ given $Cost$ should be a “soft” threshold:

![Graph showing the probability of Buys given Cost across different Cost values]

Probit distribution uses integral of Gaussian:

$$\Phi(x) = \int_{-\infty}^{x} N(0, 1)(x) dx$$

$$P(Buys? = \text{true} \mid Cost = c) = \Phi((-c + \mu)/\sigma)$$
Why the probit?

1. It’s sort of the right shape

2. Can view as hard threshold whose location is subject to noise

\[
\text{Cost} \rightarrow \text{Buys?} = \text{Cost} + \text{Noise}
\]
Discrete variable contd.

Sigmoid (or logit) distribution also used in neural networks:

\[
P(Buys? = true \mid Cost = c) = \frac{1}{1 + e^{\frac{-c+\mu}{\sigma}}}
\]

Sigmoid has similar shape to probit but much longer tails:
Bayes nets provide a natural representation for (causally induced) conditional independence

Topology + CPTs = compact representation of joint distribution

Generally easy for (non)experts to construct

Canonical distributions (e.g., noisy-OR) = compact representation of CPTs

Continuous variables ⇒ parameterized distributions (e.g., linear Gaussian)