Numerical Approximation of \((n-1)\)-Dimensional Stable Manifolds in Large Systems such as the Power System*

VAITHIANATHAN VENKATASUBRAMANIAN† and WEIJUN JI‡

Key Words—Nonlinear systems; stability; large systems; power systems.

Abstract—This paper discusses a direct method for numerical computation of the local quadratic approximations of \((n-1)\)-dimensional stable manifolds for the unstable equilibria on the stability boundary in large systems. The algorithm proposed here does not require time-consuming normal form or symbolic computations. The numerical procedure is illustrated on transient stability problems in simple power system models. © 1997 Elsevier Science Ltd.

1. Introduction

Let us consider smooth nonlinear systems of the form

\[ \Sigma: \dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f: \mathbb{R}^n \to \mathbb{R}^n \text{ is } C^r, \]

and suppose \(x_0\) is a locally asymptotically stable equilibrium point for \(\Sigma\). The region of attraction \(A\) for \(x_0\) can be defined as usual

\[ A = \{ x_0 \in \mathbb{R}^n | \Phi(x) \to x_0 \text{ as } t \to \infty \}, \]

where \(\Phi(t)\) is the flow operator for \(\Sigma\). One of the fundamental problems in nonlinear theory is the analysis of the nature of the region of attraction \(A\) and its boundary \(\partial A\). More specifically the structure of the boundary \(\partial A\) (denoted from here onwards of the stability boundary) is crucial in determining the global stability properties of the dynamics \(\Sigma\).

In power system analysis which motivates this paper, stability boundary and the region of attraction are fundamental concerns in transient stability computations where the main problem is to determine whether a "post-disturbance" initial condition lies within a suitable region of attraction. The works of Varaiya et al. (1985), Chiang et al. (1988) and Zaborszky et al. (1987, 1988) developed the theoretical foundations for tackling the dynamic stability monitoring problems by providing characterizations of the stability boundary under Morse-Smale theoretic assumptions. Specifically it was proved in Zaborszky et al. (1988) that the stability boundary is principally composed of \((n-1)\)-dimensional stable manifolds of order \(n-1\) unstable equilibrium points. Here, an equilibrium \(x_i\) is an order \(n-1\) equilibrium point if the system Jacobian at \(x = x_i\) has exactly one eigenvalue in the open right-half complex plane \(C^+\) and the remaining \(n-1\) eigenvalues in the open left-half complex plane \(C^-\).

*Received 7 August 1995; received in final form 27 March 1997. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor M. Ikeda under the direction of Editor A. P. Sage. Corresponding author Professor V. Venkatasubramanian. E-mail mansi@ecce.wsu.edu.

†School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.

‡Refer to the concept of "quasi-stability boundary" \(\partial A\) in Zaborszky et al. (1988) and its characterization.

§The notation "order \(n-1\) equilibria" in Zaborszky et al. (1988) is equivalent to the notation "type I equilibria" in Chiang et al. (1988).

The stability boundary results in Zaborszky et al. (1988) are valid under certain generic (hyperbolicity and transversality) assumptions along with a nongeneric assumption on the existence of a suitable energy function (which has been proved to be valid in relevant power system models Varaiya et al. (1985)). Therefore, it follows that the stability boundary properties can be well understood by computing the stable manifolds of order \(n-1\) equilibrium points. However, the main difficulty in the computation of stable manifolds is that they are very large dimensional invariant manifolds which are only defined implicitly as a collection of system trajectories. In the mathematical literature, numerical computation of stable (and unstable) manifolds has been pursued by several researchers motivated by their applications in global bifurcation analysis. The early works of Ushiki (1980) and Hassard (1980) which are extensions of Poincaré's classical results on normal-form theory propose explicit algorithms for this purpose by computing the relevant coefficients in power series representations. Ushiki (1980) derives global formulae for stable manifolds under the stronger assumption of analyticity on the function \(f\) in \(\Sigma\) while the computations in Hassard (1980) are purely local.

In the engineering context, the fundamental paper Salam et al. (1983) proposed explicit formulae for computing the Taylor series coefficients of stable manifolds at unstable equilibria by Poincaré’s normal form like expressions (see Salam et al. (1983), Theorem 2). On the other hand, the paper Zaborszky et al. (1987) proposes an alternate approach for computing the terms in the Taylor series expansion of stable manifolds through symbolic manipulations. While both these efforts (Salam et al. 1983, Zaborszky et al. 1987) were aimed at approximating the stability boundary, each method has some limitations when applied to large system models. The normal form like computation of the stable manifolds in Salam et al. (1983) (see also the recent work of Saha et al. (1995)) require knowledge of all the eigenvalues and the eigenvectors at an order \(n-1\) unstable equilibrium which is quite computationally-intensive. The symbolic approach in Zaborszky et al. (1987) is also difficult to implement in general large-scale nonlinear system models.

In this paper, we propose a more straightforward procedure for computing the quadratic approximation of the \((n-1)\)-dimensional stable manifold at an order \(n-1\) equilibrium point by extending a method proposed by us recently in Ji and Venkatasubramanian (1995) for the computation of center manifolds. The spirit of our computational method in power series expansions of invariant manifolds as solutions of certain linear Sylvester control theory and this concept has been well known in nonlinear control theory since the 1960s (Aitken, 1961). The contribution of our work is the derivation of explicit formulae for the quadratic component of the \((n-1)\)-dimensional stable manifold which are also easy to implement in large systems. The methods of the paper can also be extended for computation of unstable manifolds. However, we focus on stable manifolds in this paper since our main interest is in approximation of stability boundary which is in turn composed of \((n-1)\)-dimensional stable manifolds.

The formulae and the computational method are presented in Section 2. In Section 3, we illustrate the method on stability boundary analysis in small-scale power system models.
2. Quadratic local approximation of the stable manifolds

In this section, we restrict the discussion to the quadratic approximation of the stable manifold at an order \( n - 1 \) equilibrium point. Without loss of generality, it is assumed that the origin is the saddle point under consideration.

Let us rewrite \( \Sigma \) in the form

\[
\dot{x} = Ax + R(x),
\]

where \( A \) is the system Jacobian matrix at the origin which has one eigenvalue in the open right-half plane \( \mathbb{C}^+ \) and \( n - 1 \) eigenvalues in the open left-half plane \( \mathbb{C}^- \) and \( R(x) \) denotes the higher (\( \geq 2 \)) order terms of the system with \( R(0) = 0, D(R(x))_{|x=0} = 0 \). By the Stable Manifold Theorem (Smale, 1967), it follows that the equilibrium point \( x = 0 \) has a unique one-dimensional unstable manifold say \( W^u \) and a unique \( (n - 1) \)-dimensional stable manifold say \( W^s \). Our interest here is the computation of the stable manifold \( W^s \) locally in suitable chart coordinates.

As a first step, we need to find a linear transformation which converts the above system (3) into a convenient form (4) and (5) shown below so that the stable (s) and unstable (u) dynamics are decoupled in the linear sense,

\[
\dot{u} = A_s u + R_s(u, v), \quad u \in \mathbb{R}^{n-1}, \tag{4}
\]

\[
\dot{v} = A_v v + R_v(u, v), \quad v \in \mathbb{R}^1, \tag{5}
\]

where the matrix \( A \) has all its \( n - 1 \) eigenvalues with negative real parts and \( \text{Re}(\lambda_s) > 0 \).

It is obvious that the linear transformation that we seek is easy to construct when all the generalized eigenvectors of the system are available. In this case we simply let

\[
Q = (v_1, v_2, \ldots, v_n),
\]

where \( v_i \) for \( i = 1, \ldots, n - 1 \) are the generalized eigenvectors of matrix \( A \) in \( \mathbb{C}^- \) and \( v_n \) is the eigenvector corresponding to the eigenvalue \( \lambda_s \in \mathbb{C}^- \). Clearly, the transformed system matrix \( Q^{-1}AQ \) will render the linear part into the Jordan canonical form so that the stable and unstable eigenspaces are nicely decoupled. However, computation of all the generalized eigenvectors in large-scale systems like in practical power system models is extremely computation intensive and may even be impossible in truly large-scale problems. On the other hand, there exist efficient algorithms for the computation of the "dominant" unstable eigenvalue \( \lambda_s \) and its corresponding eigenvector say \( v_n \) in large-scale systems (for instance, refer to the algorithms in Zaborszky et al. (1995) which use the bilinear transformation and matrix power methods for efficient computation of the critical eigenvalue and the associated eigenvectors).

The linear transformation matrix \( Q \) for converting the system representation from (3) to (4) and (5) by \( x = Q^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \) can be easily constructed using only the information on the eigenvector \( v_n \) of the unstable eigenvalue as is shown in the following proposition.

**Proposition 1.** Suppose the matrix \( A \) has one eigenvalue \( \lambda_s \in \mathbb{C}^- \) with associated eigenvector \( v_n \) and all other eigenvalues in \( \mathbb{C}^+ \), then there exists a linear transfer matrix \( Q \) such that

\[
Q^{-1}AQ = \begin{pmatrix} A_s & 0 \\ 0 & \lambda_s \end{pmatrix},
\]

where \( Q = Q_1 Q_2 \) and the matrices \( Q_1 \) and \( Q_2 \) are constructed as follows:

(i) Re-scale \( v_n \) by its largest absolute value element such that

\[
v_n = \begin{pmatrix} q_s^{-1} \\ 1 \\ q_v^{-1} \end{pmatrix},
\]

where \( q_s^{-1} \) and \( q_v^{-1} \) are \((i - 1) \times 1\) and \((n - i) \times 1\) vectors, respectively with all its elements having absolute values not greater than 1. Then we have

\[
Q_1 = \begin{pmatrix} I_{i-1} & 0 & q_v^{-1} \\ 0 & 0 & 1 \\ 0 & I_{n-i} & q_s^{-1} \end{pmatrix},
\]

so that

\[
P_1 = Q_1^{-1} = \begin{pmatrix} I_{i-1} & -q_v^{-1} & 0 \\ 0 & -q_s^{-1} & I_{n-i} \end{pmatrix}
\]

(ii) Solve for \( Q_{12} \) from

\[
(A_s - A_{21})Q_{12} = -A_{21},
\]

where \( A_s \) and \( A_{21} \) are obtained from the product

\[
P_1 A Q_1 = \begin{pmatrix} A_s & 0 \\ A_{21} & \lambda_s \end{pmatrix}
\]

Construct the matrix

\[
Q_2 = \begin{pmatrix} I & 0 \\ Q_{12} & 1 \end{pmatrix}
\]

so that

\[
P_2 = Q_2^{-1} = \begin{pmatrix} I & 0 \\ -Q_{12} & 1 \end{pmatrix}
\]

and we have

\[
Q = Q_1 Q_2, \quad P = Q^{-1} = P_2 P_1.
\]

The proof of the Proposition directly follows by standard linear algebraic arguments and is omitted. We remark that the above equations being straightforward linear matrix manipulations are readily suited for sparsity techniques in large systems and hence, the computational effort involved in the above algorithm is much smaller compared to the computation of all the generalized eigenvectors. For instance, when the matrix \( A \) is sparse, it can be seen from Proposition 1 that the matrices \( P_1, Q_1, P_2 \) and \( Q_2 \) are sparse by construction; the matrix \( A_s \) is sparse as a product \( P_1 A Q_2 \) of three sparse matrices; and the computation of \( Q_{12} \) follows by sparsity techniques. Therefore, the computation of the linear transformation matrix \( Q \) is straightforward and, moreover, there exists an added benefit that the inverse matrix \( P = Q^{-1} \) is also automatically computed within this algorithm.

2.1. Quadratic components. Suppose now that the linear portion of the local dynamics has been decoupled so that the system has been represented for the form (4) and (5), then by the Stable Manifold Theorem, the stable manifold \( W^s \) exists and can be represented uniquely by a graph of the form

\[
v = h(u),
\]

locally near the origin. The invariance of the stable manifold implies that \( W^s \) or equivalently the graph \( v = h(u) \) must satisfy the PDE

\[
\dot{\lambda}_s h(u) + R_s(u, h(u)) = \delta h(\dot{A}_s u + R_s(u, h(u))), \tag{6}
\]

with \( h(0) = 0 \) and \( D h(0) = 0 \).

For computing the quadratic approximation of the stable manifold \( W^s \), we let

\[
v = h(u) = u^T H_{22} u + 0(3), \tag{7}
\]

where \( H_2 \) is a \((n - 1) \times (n - 1)\) matrix which completely characterizes the quadratic local approximation of the stable manifold. Therefore, our problem reduces to computing the entries of the \( H_2 \) matrix. This can be easily carried out as follows.

Let \( O_2(x) \) denote the \( rth \) order Taylor series expansion terms of \( f(x) \), then by setting \( r = 2 \), the equation for computing the quadratic stable manifold can be simplified into

\[
\delta h(\dot{A}_s u + R_s(u, h(u))) = 0.
\]
Let us denote
\[ O_2(R_d(u,0)) = u^TGu \] (8)
and the computational method for evaluating the matrix \( G \) will be discussed in the next subsection. Then for the quadratic case the invariance condition reduces to
\[ \lambda u^TH_1u + u^TGu - 2a^TH_2Au = 0. \]

Thus, \( H_2 \) can be solved from the linear matrix equation
\[ H_2(\lambda I - 2A_d) = -G. \] (9)

Note here that the matrix \((\lambda I - 2A_d)\) is always invertible (and is sparse) hence, the quadratic stable manifold matrix \( H_2 \) can always be solved uniquely from (9). Therefore, for the quadratic stable manifold approximation, what remains is the computation of the \( G \) matrix which consists of certain second-order coefficients of the \( \nu \) dynamics.

2.2. Computation of the \( G \) matrix. By definition of the \( G \) matrix (8), computing \( G \) is quite simple from (5) if the vector field is already in the decoupled form (4) and (5). However, in general problems, the original system (3) requires the linear transformation \( Q \) before appearing in the decoupled form (4) and (5) which then renders the computation of \( G \) quite messy in the original \( \nu \)-coordinates. In this section, we propose a systematic way to evaluate \( G \) in the original system coordinates \( x \) using straightforward numerical differencing type algorithms. The procedure is summarized as a Proposition.

**Proposition 2.** Express the original vector field into scalar form as
\[ \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n), \]
\[ \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n), \]
\[ \ldots \]
\[ \dot{x}_n = f_n(x_1, x_2, \ldots, x_n). \]

Suppose \( x = 0 \) is the equilibrium point of the system and the linearized matrix at \( x = 0 \) has one eigenvalue in \( \mathbb{C}^+ \) and all other eigenvalues in \( \mathbb{C}^- \). Let \( \mathcal{Q} \) denote the matrix that decouples the linear part of the vector field into the \((n-1)\)-dimensional stable eigenspace \( s \) and the one-dimensional unstable manifold \( \nu \) in (4) and (5). Suppose the \( nth \) row of \( P = Q^{-1} \) matrix is
\[ (p_{n1}, p_{n2}, \ldots, p_{nn}). \]

Define the new scalar function
\[ s(x) = \sum_{i=1}^{n} p_{ni}f_i(x). \]

and let \( A_2 \) be the second-order Hessian matrix associated with \( s(x) \), i.e.,
\[ A_2 = \begin{bmatrix} \frac{\partial^2 s}{\partial x_1^2} & \ldots & \frac{\partial^2 s}{\partial x_1 x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 s}{\partial x_n x_1} & \cdots & \frac{\partial^2 s}{\partial x_n^2} \end{bmatrix} \bigg|_{x=0}. \]

Then the matrix \( G \) in the quadratic stable manifold computation can be obtained as the upper left \((n-1) \times (n-1)\) sub-matrix of the matrix \( G_\nu \) where
\[ G_\nu = \frac{1}{2}Q^TA_2Q. \]

**Proof.** Consider the Taylor series expansion of the vector field at \( x = 0 \) up to order 2
\[ \dot{x} = Ax + \frac{1}{2} \begin{bmatrix} x^T \frac{\partial^2 f_i(x)}{\partial x^2} |_{x=0} \end{bmatrix} + O(3). \]

Let \( x = Q(u,\nu) \) be such that
\[ Q^{-1}AQ = \begin{pmatrix} A_d & 0 \\ 0 & \lambda_\nu \end{pmatrix}, \]

then
\[ \dot{\nu} = \dot{\lambda}_\nu + \frac{1}{2}(u,\nu)^TQ^T \sum_{i=1}^{n} \frac{\partial^2 f_i(x)}{\partial x^2} \bigg|_{x=0}Q\left(\begin{array}{c} u \\ \nu \end{array}\right) + O(3). \]

Now, since the Jacobian matrix \( A_d \) and \( \lambda_\nu \) are available as discussed earlier in Section 2, the computation of \( H_2 \) can proceed as discussed in Section 2.1.

3. Algorithm for computing the quadratic stable manifold

To summarize the discussion thus far in the paper, we state the following algorithm for the local approximation of the quadratic stable manifold at an order \( n-1 \) saddle point say \( x_0 \).

(i) Compute the unstable eigenvalue \( \lambda_\nu \) and the associated eigenvector \( \nu_\nu \) at \( x_0 \). Re-scale \( \nu_\nu \) by dividing its largest absolute value element (suppose it is the \( i \)th element) so that \( \nu_\nu \) has the form
\[ \nu_\nu = \begin{pmatrix} 0^{i-1} \\ 1 \\ 0^{n-i} \end{pmatrix}. \]

Construct
\[ Q_1 = \begin{pmatrix} I_{i-1} & 0 & 0^{i-1} \\ 0 & 1 & 0^{i-1} \\ 0 & I_{n-i} & 0^{n-i} \end{pmatrix} \]

and
\[ P_1 = Q_1^{-1} = \begin{pmatrix} I_{i-1} & q_1^{i-1} & 0 \\ 0 & q_1^{i-1} & I_{n-i} \\ 0 & 1 & 0 \end{pmatrix} \]

Compute \( A_2 \) and \( A_{21} \) from the product
\[ P_1AQ_1 = \begin{pmatrix} A_d & 0 \\ 0 & \lambda_\nu \end{pmatrix}. \]

(ii) Compute \( Q_{12} \) from
\[ (\lambda I - A_d)Q_{12} = -A_{21} \]

and construct the matrices
\[ Q_2 = \begin{pmatrix} I & 0 \\ Q_{12} & I \end{pmatrix} \]

and
\[ P_2 = Q_2^{-1} = \begin{pmatrix} I & 0 \\ -Q_{12} & I \end{pmatrix}. \]

Compute the matrices
\[ Q = Q_1Q_2, \quad P = P_2P_1, \]

which completes the construction of the linear transformation matrix \( Q \) and its inverse \( P \).

(iii) Let
\[ s(x) = \sum_{i=1}^{n} p_{ni}f_i(x) \]

and evaluate \( A_3 \) as the second-order Hessian matrix associated with \( s(x) \) at \( x_0 \), i.e.,
\[ A_3 = \begin{bmatrix} \frac{\partial^2 s}{\partial x_1^2} & \ldots & \frac{\partial^2 s}{\partial x_1 x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 s}{\partial x_n x_1} & \cdots & \frac{\partial^2 s}{\partial x_n^2} \end{bmatrix}. \]
Note that the Hessian matrix $A_2$ can be computed by either numerically differencing (twice on the function $s(x)$) or by numerical differencing once on the associated Jacobian matrix if so available. Next compute the matrix

$$G_r = \frac{1}{2}Q^T A_2 Q.$$  

Then the matrix $G$ in the quadratic stable manifold computation can be obtained as the upper left $(n - 1) \times (n - 1)$ sub-matrix of $G_r$.

(iv) Compute $H_2$ from

$$H_2(x_{i-1}) = -G_r,$$

or equivalently from

$$H_2 = -G(x_{i-1} - 2A_2)^{-1}.$$

(v) Then the local quadratic stable manifold approximation at $x_a$ is given by

$$v = u^T H_2 u_k,$$

with

$$\begin{pmatrix} u_k \cr v_k \end{pmatrix} = P(x_i-x_{i-1}).$$

The algorithm proposed above for the computation of the linear transformation matrix $P$ and the quadratic component $H_2$ can be used in assessing whether an initial condition, say $x_0$, is within the region of attraction $A$ of a stable equilibrium point $x_e$. As stated in the Introduction, we will assume that (Zaborszky et al., 1988): (1) Morse-Smale assumptions are in effect; (2) the saddle point $x_s$ lies on the quasi-stability boundary $\partial A$. Note that it is not just sufficient to assume that the saddle point $x_s$ lies on the stability boundary $\partial A$ because even when $x_s \in \partial A$, there exists a possibility that both "sides" of the stable manifold belong to the region of attraction $A$ (see Fig. 1) which is similar to the example in Zaborszky et al. (1988a). However, this possibility is ruled out by the stronger assumption $x_s \in \partial A$ since then it follows that $W(x_s) \subset \partial A$ (Zaborszky et al., 1988) and the stable manifold $W(x_s)$ indeed separates the initial conditions within the stability region (or the quasi-stability region $\text{int}(\hat{A})$) to be more precise) from those outside the stability region.

Note that in Fig. 1, $x_s \in \partial A$, however both sides of $W(x_s)$ belong to the region of attraction. This is because in this case $x_s \in \text{int}(\hat{A})$ and $x_s \not\in \partial A$. Hence, $W(x_s) \subset \text{int}(\hat{A})$ (Zaborszky et al., 1988) and the stable manifold $W(x_s)$ in Fig. 1 should not be considered as part of the stability boundary computations as in this paper. In other words, in Fig. 1, even though $W(x_s)$ is part of the stability boundary $\partial A$, it should not be considered as stability boundary for practical purposes since $W(x_s)$ belongs to the quasi-stability region $\text{int}(\hat{A})$ which is the whole space $\mathbb{R}^2$ in this example. More details on the concept of quasi-stability boundary can be seen in Zaborszky et al. (1988, 1988a).

Under assumptions (1) and (2), since the stable manifold locally provides an excellent approximation of the stability boundary, given an initial condition $x_0$ "near" the saddle point $x_s$, the straightforward procedure stated below can be used for determining whether the initial condition $x_0$ lies inside the stability boundary (hence, within the stability region) or outside the stability boundary (hence, not within the stability region).

(i) Compute the transformed vectors

$$\begin{pmatrix} u_k \cr v_k \end{pmatrix} = P(x-x_{i-1}),$$

or equivalently from

$$\begin{pmatrix} u_k \cr v_k \end{pmatrix} = P(x_0-x_{i-1}).$$

(ii) If the sign of $v_k - u_k^T H_2 u_k$ agrees with the sign of $x_0 - x_{i-1}^T H_2 x_{i-1}$, then the initial condition $x_0$ lies within the region of attraction of $x_s$. If the two signs disagree then $x_0$ does not belong to the region of attraction of $x_s$.

Note that this procedure is straightforward, hence, is easy to implement in large-scale systems. The validity of the approach depends on the size of the deviation $x_0-x_{i-1}$ of the initial condition $x_0$ from the saddle point $x_s$ and for instance an a priori chosen upper bound on the norm of the deviation $x_0-x_s$ can be used to limit the applicability of this procedure.

4. Stability boundary analysis in power system models

In this section, we apply the algorithm proposed for stable manifold approximation in Section 3 on stability boundary analysis in small-scale power-system models. The strength of our approach is in the simplicity of the algorithm as stated in Section 3, which makes it a powerful procedure both from analytical and implementation view-points in large-scale models. However, the emphasis in this section is on illustration of the method through several small-scale models.

4.1. Geometry of the stability boundary. As noted in the Introduction, under certain Morse-Smale like assumptions, the stability boundary of the operating point for the power system can be proved to mainly consist of the $(n-1)$-dimensional stable manifolds of the unstable equilibria on the quasi-stability boundary Zaborszky et al. (1988). This fact naturally allows us to draw an insight on the global stability properties of the power system dynamics by computing the Taylor series expansions of the stable manifolds on the stability boundary as was noted in Zaborszky et al. (1987). The recent work of Saha et al. (1995) specifically uses the quadratic information of the stable manifold at a type one saddle-point for studying the relationship between

![Fig. 1. Qualitative phase-portrait of a nonlinear system.](image-url)
system stress and the geometry of the stable manifold aimed at assessing the "critical" physical-state variables which influence the dynamic security monitoring type algorithms. In this section, we use the algorithm developed in Section 3 for computing geometric approximations of the stability boundary.

**Example 1.** Consider the classical swing equation model of the single-machine-infinite-bus power system model

\[ \Sigma_1: \quad \delta = 2n_0\delta, \quad (10) \]

\[ \dot{\delta} = \frac{1}{2H} \left( -D\delta + P_T - \frac{E}{\delta} \sin \delta \right). \quad (11) \]

The parameters \( H = 5, E' = 1.1, \delta_x = 0.25 \) and \( x = 0.25 \) are assumed to be constant while the parameters \( P_T \) and \( D \) are varied as shown in Fig. 2 for drawing some insight on the structure of the stability boundary under power \( P_T \) and damping \( D \) variations.

In the \( \delta-\omega \) phase-portraits shown in Fig. 2, the solid loci represent the quadratic approximations of the stable manifolds computed using the algorithm in Section 3, whereas the bold dotted loci correspond to the actual stability boundary. By comparison of the two loci, it is easily seen that the quadratic approximations are valid only locally near the saddle points even in this very simple power-system model, even though the region of validity of the quadratic approximation looks quite healthy (with angle \( \delta \) deviations about \( \pm 1 \) radian around the saddle point) in all the simulations. Moreover, interestingly the quadratic approximation is a conservative local estimate of the stability boundary on one-side and is an optimistic estimate of the boundary on the other side of the saddle point under consideration.

**4.2. Approximate computation of critical clearing times.** In this section, we demonstrate an application of the stability boundary approximation method for computing critical clearing times in power system transient-stability analysis. Computation of the critical clearing time follows by evaluating at different time-instants along the fault-on trajectory whether an initial condition lies inside or outside the region of attraction of the post-fault operating point. The geometric procedure stated at the end of Section 3 can then be readily used for computing the critical clearing times provided the fault-on trajectory stays sufficiently close to the saddle point where the stable manifold approximations are computed. The algorithm as implemented in this section assumes that the critical saddle point \( x_s \) has been previously located using other computational methods. This assumption on \( x_s \) is a conventional assumption in power system transient-stability methods and standard algorithms such as the controlling UEP (unstable equilibrium point) type transient energy function methods Fould and Vittal (1991) include in them excellent procedures for locating the saddle point \( x_s \) under consideration.

**Example 2.** Let us consider the single-machine-infinite-bus power system in Fig. 3 and consider a solid 3-phase line-to-ground fault in the middle of the lower transmission line. The parameters of the system are chosen as \( M = 10, f_0 = 60, E' = 1.1, x = 0.5, x_d = 0.25, D = 5, P_T = 1.2 \). The phase-portrait for the post-fault system (with the lower line open) is shown in Fig. 4. The fault-on trajectory is shown in Fig. 4 by the solid line with the cross-marks while the quadratic stable manifold approximation is plotted by a solid locus. The actual stability boundary is shown by the bold dotted locus as in Fig. 2. Note that in this example, indeed the actual stability boundary (bold dotted locus) agrees very well with the approximate stability boundary (solid locus) near the critical clearing segment of the fault-on trajectory. Actual computations yield the critical clearing time to be \( t_c = 0.2822 \text{s} \) (approximately 17 cycles) with the

![Fig. 3. A single-machine-infinite-bus power system.](source)

**Fig. 2.** State space phase-portraits for the classical SMIB system.
actual boundary while we get $t_c = 0.2893$ (approximately 17 cycles) using the quadratic approximation of Section 3. Indeed both agree well even though the quadratic approximation is somewhat optimistic. For comparison, the linear approximation of the boundary can also be computed which gives the clearing time to be $t_c = 0.3332$ (approximately 20 cycles). Clearly, the quadratic approximation is closer to the exact $t_c$ value compared to the linear approximation.

Example 3. Consider the multi-machine model shown in Fig. 5 and all the generators here are modeled by classical swing equation representations. The loads are modeled by constant impedance representations so that network reduction can be used for the power system in Fig. 5. The angle dynamics of the reduced system can now be modeled by four differential equations as usual. The parameter values are assumed to be $E_1 = 1.25, E_x = 1.3, H = 1, D = 2, H = 3.7699, D_2 = 1.8850, P_R1 = 1$ and $P_R2 = 1$. Consider a solid fault at the middle of the transmission line connecting buses 1 and 3. Routine computation yields $t_c = 0.4417$ s (roughly 26 cycles) for this fault when the actual stability boundary is considered. Next when we use the approximate stability boundary, the critical clearing time is computed to be $t_c = 0.4412$ s (roughly 26 cycles again) which clearly matches very well the actual $t_c$ value.

Example 4. Consider the power-system configuration as in Example 2 with the same fault but with a more detailed generator model. This time the generator voltage dynamics is modeled by the single-axis flux decay model (damper windings are assumed to be not present) including an excitation control model (IEEE-type DC1A exciter), so that the overall power system model can be written as

$$
\delta = 2\pi f(\omega),
$$

$$
M\dot{\delta} = -D\delta + P_T - \frac{1}{x_d + x}(E'\sin\delta - E\cos\delta),
$$

$$
T_{\omega}\dot{\omega} = - \frac{x_d + x}{x_d + x}E + \frac{x_d - x_d}{x_d + x}\cos\delta + E'\omega,
$$

$$
T_A\dot{V}_A = -V_A + K_d(V_{ref} - V - V_f), \quad V_{\text{min}} \leq V_A \leq V_{\text{max}},
$$

$$
T_E\dot{E}_E = -(K_EE + AE'\omega + V_A),
$$

$$
T_F\dot{F} = -V_k + K_FE_E.
$$

where $V$ is the bus voltage

$$
V = \frac{1}{x + x_d}\sqrt{(x_d)^2 + x^2E^2 + 2xx_dE\cos\delta},
$$

and the parameters are chosen to be

$$
M = 6.1, \quad P_T = 0.8, \quad x_d = 0.475 \quad D = 1,
$$

$$
T_{\omega} = 5.33, \quad T_E = 1.15, \quad T_A = 0.89 \quad T_F = 0.62,
$$

$$
K_E = -0.1, \quad K_A = 20, \quad A = 0.0027 \quad B = 0.75,
$$

$$
V_{\text{ref}} = 1.05, \quad K_F = 0.058, \quad x = 0.25.
$$

Then direct integration method gives clearing time $t_c = 0.419$ (25 cycles), whereas the approximate method given in this paper gives clearing time $t_c = 0.473$ (28 cycles). It seems from our preliminary simulations that the actual clearing times agree very well with those computed using stable manifold type approximation method when only the swing equations are modeled in the classical multi-machine model. However, the stable manifold
type methods yield somewhat optimistic results (which may be undesirable from a practical view point) for the critical clearing time when more detailed generator model are considered as in Example 4.

The main advantage of the algorithm proposed in this paper is that the algorithm is straightforward even in very detailed power system models. The preliminary simulations indicate that the stable manifold based method proposed in this paper for computing the critical clearing time is relatively much faster compared to the numerical integration type methods. However, at this point, the main limitation in the stable manifold approximation type algorithm is that this method is applicable only when the fault-on trajectory (or the critical clearing state) happens to be near the saddle point $x_s$ used in the algorithm. More detailed algorithms which extend the local stable manifold approximations to global domains in the state space will be presented elsewhere. Further analysis is indicated on the accuracy of the algorithm and the implementation strategies of the proposed techniques in detailed large-scale power system models.

5. Conclusions

This paper proposes a simple method for computing quadratic approximations of the stable manifolds on the stability boundary. Geometric algorithms are developed for testing whether an initial condition that is "near" a saddle, in fact, lies within the region of attraction or outside. The algorithm is valid under certain Morse-Smale like assumptions and is easy to implement in large-system models. Power system examples used for illustrating the algorithms provide promising results on stability monitoring type analysis. The basic algorithm presented here can be readily extended for computing the stable manifolds of unstable periodic orbits and can also be modified to accommodate differential-algebraic models. Further research is indicated on the problem of global representations of the stable manifolds in large-scale models.

Acknowledgements—This research was partially supported by NSF grants ECS-8457126 and ECS-9220041. Support from Bonneville Power Administration, Portland, OR and Union Electric Company, St. Louis, MO are also gratefully acknowledged.

References


