Voltage Dynamics: Study of a Generator with Voltage Control, Transmission, and Matched MW Load
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Abstract—A comprehensive analysis of the dynamic behavior for a rudimentary but representative model of the power system is carried out when control gain and load are varied as parameters. The voltage dynamics model is subject to algebraic constraints in the form of load flow equations and is studied here as a differential-algebraic system in state and parameter spaces. Singularities in the state space (noncausal points) and bifurcations in the parameter space are the principal and interacting structural elements. A rich structure of bifurcations emerges which is analyzed here. Both local bifurcations (e.g., saddle node, Hopf) and global bifurcations (e.g., saddle connection) play vital roles. Some of the bifurcations are directly connected to the singularity and are studied here for the first time (the singularity induced bifurcation and the singularity connection bifurcation). A boundary composed of pieces of the saddle node, Hopf, and singularity induced bifurcation manifolds limits the region in the parameter space, the feasibility region, where practical operation at a specific stable equilibrium is possible. The mathematical analysis is facilitated by singularly rescaling time, which transforms the differential algebraic system into a smooth dynamic system. In the state space, the characteristics of stability boundaries are observed and a description of the regions of attraction of all equilibria are given. Stability boundaries may also include singular sections, and special singular points, pseudo equilibria, also become additional anchor points for stable and unstable manifolds. The loosely understood term of voltage collapse is classified into well defined types on both the dynamic and parametric sides.

I. INTRODUCTION

The global dynamic behavior of voltage over the practical operating range of a power system under both normal and emergency conditions is very complex. The state space contains regions of attractions of stable equilibria and other regions where stable operation is not possible. Typically, the parameter space breaks up into several open, connected regions separated by boundaries corresponding to particular behavior patterns of the system in the state space. The topology of the state space changes as parameters move across the boundary from one open region into another. The boundaries between the regions correspond to bifurcation phenomena.

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Specifc of this decomposition are poorly understood. Only recently has the significance of the role which is played by the boundaries between different regions of structurally stable behavior in the parameter space been realized (e.g., [1], [2]). The bulk of today’s literature still seems to be devoted to the study of the saddle-node bifurcation, which occurs at the tip of the PV curve. Not always identified as a bifurcation, this phenomenon has been studied for a long time (for example, it is described and analyzed in one author’s 35 year old textbook [3]). A few papers dealt with the Hopf bifurcation which is related to the creation of limit cycles in the state space ([4]–[6]). Saddle node and Hopf bifurcations are the most commonly studied local codimension one bifurcations in the power system literature. Because of the multitude of parameters present in the system, further bifurcation types will occur. For instance, a homoclinic orbit was observed as part of a simulation study in the aftermath of a Hopf bifurcation in [6]. Here, this particular phenomenon will be explained as the occurrence of a saddle connection bifurcation in the vicinity of a local codimension two bifurcation when Hopf and saddle-node bifurcations merge. Furthermore, power system dynamics are typically constrained by load flow equations resulting in a mathematical model which is differential algebraic. As a consequence, the dynamic system has singularities. The great importance of these singularities has been pointed out by Zaborszky in [7], by Hill, Hiskens, and Mareels (impasse surfaces) in [8]–[10], and was rigorously analyzed by us in [11]. As will be seen below, singularities can act as stability boundaries, be sources or sinks of trajectories and cause further bifurcations. The presence of some of these singularity related bifurcations has been noted in the power system literature recently [1], [7], [12]–[14] and we give a rigorous analysis here for a rudimentary system.

No comprehensive study of voltage stability which takes into account the effect of a variety of bifurcations exists in the literature. Here, such a comprehensive study of a third-order rudimentary model for the power system will be established, greatly expanding earlier work at Washington University [7], [15]. While the model has to be quite simplified to be mathematically tractable, it retains the rudiments of the four principal components of the power system which affect the stability, namely the generator, voltage control, transmission, and the load. In this paper, a comprehensive analysis for variations of control gain
and loading will be given and correlated to the dynamic behavior in the state space. Even in this low order, an amazingly rich collection of practical behavior patterns emerges. In fact, the interacting structure observed in the state space and the parameter space for this rudimentary system is preserved even for more general differential algebraic systems as is shown in [11]. The new concepts introduced here such as singularity induced bifurcation and feasibility region are not restricted to this rudimentary model. However, the low degree of the system in this paper has the advantage that results can be readily displayed graphically, fostering both understanding and insight.

The dynamic behavior in the state space is studied here mostly through phase portraits rather than the more conventional time histories for individual states. The phase portrait is interpreted in a general and multidimensional sense as the flow of a vector field, i.e., as the totality of all integral curves. For this rudimentary system, the phase portraits are directly visualizable in the plane. Hence, concepts such as instability and damping of trajectories can easily be observed and the regions of attractions and the regions of voltage collapse can be identified.

The paper is organized as follows: Section II gives a mathematical description of the model; a physical-mathematical description of the observed phenomena follows in Section III, while the mathematical foundations are developed in Section IV.

II. A MODEL OF THE RUDIMENTARY POWER SYSTEM WITH GENERATOR, VOLTAGE CONTROL, TRANSMISSION, AND MATCHED LOAD

Voltage dynamics for the rudimentary case of a generator with load at the end of a transmission line as shown in Fig. 1 will be considered. In some cases, SVC control on the load is also explored instead of the excitation control. Since there is only one inertia in this system, the assumption \( P = P_d \) with no initial stored energy implies that \( \dot{\theta} = 0 \) and \( \theta = \text{constant} \). Thus the voltage dynamics is isolated from the angle dynamics. So this assumption is a counterpart for voltage dynamics of the so-called “classical” model which isolates the rotor angle dynamics by assuming constant voltage behind the transient reactance (i.e., \( E^* = 0 \)). Using a simplified one-axis generator model and a first-degree control dynamics, which is a simplification of the IEEE Type I excitation dynamics [16], the model for this rudimentary case can be stated as follows:

\[
T_d \dot{E} = \frac{x_d}{x_d^*} E^2 - \frac{x_d - x_d E_G \cos(\delta_G - \delta')} + E_{fd} (1)
\]

\[
T \dot{E}_{fd} = -\left( E_{fd} - E_{fd}^0 \right) - K \left( E_G - E_r \right) (2)
\]

\[
0 = \frac{EE_G}{x'} \sin(\delta_G - \delta') + \frac{E_G E}{x} \sin(\delta_G - \delta) (3)
\]

Excitation control (2) will be replaced in some instances by an equation for an SVC (a thyristor controlled reactance) at the load of the form

\[
T\dot{B}_s = -B_s + K (E - E_r) (8)
\]

and then \( Q \) can be redefined as

\[
Q = Q_0 + HE + BE^2 + B_s E^2. (9)
\]

Note that the load model includes an algebraic dependence of the reactive load on voltage in the form of (7). This either implies a combination of a power source \( Q_0 \), a current source \( HE \), and an impedance load \( BE^2 \), or acknowledges the fact that the reactive load will be a smooth nonlinear function of the voltage as was shown experimentally [17]. Then a second-degree expansion in \( E \) would be an appropriate approximation as given in (7). The quantity \( E_r \) is the set-point voltage, \( E_{fd}^0 \) is the nominal field excitation, and \( T \) and \( K \) are control coefficients. The other notations are conventional.

Note that the angle variables always appear in the form \( \delta_G - \delta \) or \( \delta_G - \delta' \) in (1)–(6) and hence using algebraic manipulations the model can be simplified as follows:

**Dynamic Equations:**

\[
\dot{E}' = \frac{1}{T_{rd}} \left[ -\frac{x + x_d}{E'} \dot{E}' + \frac{x_d - x_d}{x'} \left( \dot{E}^2 + \dot{E}' \dot{Q} \right) + E_{fd} \right] (10)
\]

\[
\dot{E}_{fd} = \frac{1}{T} \left[ -\left( E_{fd} - E_{fd}^0 \right) \right]
\]

\[
0 = \frac{E'E_G}{x'} \sin(\delta_G - \delta') + \frac{E_G E}{x} \sin(\delta_G - \delta) (3)
\]

\[
-K\left(\frac{1}{E} \sqrt{(xP)^2 + (xQ + E^2)^2} - E_r \right). (11)
\]
Algebraic Equation:
\[
0 = E^2E^2 - (x'P)^2 - (x'Q + E')^2. \tag{12}
\]

The general form of the model is therefore a system of ordinary differential equations subject to algebraic constraints.

\[
\dot{x} = f(x, y, p) \tag{13}
\]
\[
0 = g(x, y, p). \tag{14}
\]

Here \( f \) and \( g \) are, respectively, \( n \) and \( m \)-vectors of smooth, for the power system actually real-analytic, functions; \( x \) and \( y \) are, respectively, the dynamic and instantaneous state variables which span the state space \( \mathcal{X} \times \mathcal{Y} \).

For the rudimentary system (Fig. 1) \( x = (E', E_{fd}) \), \( y = (E) \), \( p \) denotes parameters and (secondary) controls which jointly span the parameter space \( \mathcal{P} = \mathcal{P}_p \cup \mathcal{P}_r \) where \( \mathcal{P}_p = \{ P, Q_0, H, B, K, T, E \} \) are respectively the system parameters and the operating parameters.

Equation (12) defines the admissible points in the state-parameter space for the power system. These are located in the algebraic or load flow locus \( 0 = g(x, y, p) \) (a manifold for this simple case) for given parameters \( p \).

Load flow is used here in a general sense since voltages like \( E \), \( E_{fd} \) may be present beside bus voltages \( E_B \) and \( E \). The set of points \( (x, y, p) \) which satisfy \( 0 = f(x, y, p) \) is called the stationary manifold (or locus) indicating the absence of dynamic events. Points which lie on both the stationary manifold and on the load flow manifold are the equilibrium points. The state space for the rudimentary system is three dimensional with the variables \( E, E', \) and \( E_{fd} \). Accordingly, manifolds can be directly visualized. In Fig. 2, this is shown for one specific set of parameters which corresponds to what will be called Type \( \mathcal{P} \) later on. Here \( SM \) denotes the stationary manifold and \( LF \), the union of \( LF_{low} \) and \( LF_{high} \), corresponds to the load flow manifold. The load flow manifold, which here is diffeomorphic to a plane, is a vertical wall in \( E_{fd} \) for this example because \( E_{fd} \) does not appear in \( g \). Since the state is constrained within the load flow manifold, all trajectories on the load flow manifold can be displayed in a planar view by projecting into the \( (E, E_{fd}) \) subspace as displayed in Figs. 6–8.

The load flow manifold is readily computed at or around regular points where the Jacobian of \( g \) with respect to \( E \) is nonsingular \( (\partial g/\partial E \neq 0) \) and \( g = 0 \) can be solved for \( E \) by the implicit function theorem. The remaining points are called singular here. (The terminology noncausal has also been used by other authors in this context [1], [9]).

The singular points as defined are the solutions of (12) and the following relation:

\[
(1 + 3x')E^4 + Hx'(1 + 3x')E^3
+ (HQ_0E + P^2 + Q_0^2)x^2 = 0. \tag{15}
\]

It is easily seen that these equations have a unique solution, and hence, the singularity exists unless both \( P \) and \( Q_0 \) are zero, which would represent an unrealistic model.

For the special case considered later on, when \( Q = Q_0 \), a simple explicit solution for the singularity can be given as the positive square roots of

\[
E_{sing}^2 = x'P^2 + Q_0^2 \quad \text{and} \quad E_{sing}^2 = 2x' \left( Q + \sqrt{P^2 + Q_0^2} \right). \tag{16}
\]

Hence, the set of singular points is precisely the line

\[
LF_{sing} = \{ (E_{sing}, E_{fd}, E_{sing}) : E_{fd} \in \mathbb{R} \} \tag{17}
\]

denoted by \( a - a \) in Fig. 2 and in all of the Figs. 6–8. This singular line divides the load flow manifold \( LF \) into two half-spaces \( LF_{low} \) and \( LF_{high} \) (see Fig. 2), called the components [11] or causal domains [9] of the state space. Each has its own dynamics which is identified by substituting the solution of (12) within the component into (10) and (11). As has been discussed extensively in [11], trajectories cannot predictably progress across the singular line. Mathematically, the model (10)–(12) implies that trajectories enter into or emerge from singular points with infinite speed in the algebraic variable \( E \). Hence, in the vicinity of the singular point the analysis based on the quasistationary model (10)–(12) simply stops and the validity of conventional AC dynamics collapses. It is therefore apparent that the singular line, \( a - a \), plays a fundamental role in the study of voltage dynamics. In the real system, this behavior is governed by the ultrafast distributed parameter dynamics of the transmission lines and other non-AC type fast dynamics [11]. In the enlarged dynamic space \( \mathcal{X} \times \mathcal{Y} \), the load flow manifold itself becomes a dynamic feature, which then raises the question of the stability of the components. This is an unsolved problem currently because of the unconventional nature (partial differential equations with a network structure) of the model for the \( y \) dynamics. Industrial experience [18] indicates, however, that stable operation at a low voltage equilibrium on other components is possible.

Voltage dynamics (and indeed system stability) depends on properties of both the state space and the parameter space, which play equally important roles. Roughly, the position of the operating point in the parameter space determines the type and topology of state space behavior.
and the position of equilibria. State space events (Figs. 6–8) dictate the specific trajectories (i.e., time histories of $E$, $E'$ or $E_{fd}$) which come into play. Even in this simple example the operating parameter space is seven dimensional, $\mathcal{P} = \{P, Q_0, H, B, K, T, E\}$. It is realistic to assume for study purposes that the control set point is $E_c = 1$pu at the operating point and that the nominal reactive power at 1pu voltage is proportional to the real power $Q/P = q = \text{constant}$, which means constant power factor. Experiments show that performance is much more sensitive to control gain $K$ than to $T$. Accordingly, a two-dimensional parameter subspace consisting of power $P$ (or equivalently $Q$) and control gain $K$ is selected for study. This choice represents the most active and hence most informative subset of the operating parameter space. The presentation here is not challenged by exploratory variations of other parameters such as $T$ or the power factor, i.e., $q$.

At this point, the problem has been reduced to a dynamic state space $(E, E_{fd})$ and an active parameter space $(P, K)$ (equivalently $(Q, K)$ under the constant power factor assumption) which are both two dimensional and can thus be presented in a plane.

III. A Description of the Dynamics of the Model

The objective in this section is to establish the system behavior across the parameter space and the state space. Each point in the parameter space defines a load-flow manifold and a corresponding phase portrait. With increasing load, for a fixed value of the control gain, the type of operation changes as bifurcations occur. Fig. 3 gives the qualitative structure of the bifurcation diagram for the parameters $P$ and $K$. In Fig. 4 an actual simulation is given for a specific set of values. For simplifying the analysis given in Section IV, it will be assumed in the rest of the paper as in Figs. 3 and 4 that the reactive power is represented by a constant load i.e., $Q = Q_0$. However, as can be seen from (15), singular points exist as long as either $P$ or $Q_0$ is nonzero. Furthermore, the results for the constant reactive load case are not challenged by simulation even for the general composite load type. Fig. 10 shows the bifurcation diagrams for the three special load types, i.e., the reactive power is represented as a constant load $Q_0$ in Fig. 10a, constant current source $H$ in Fig. 10b and as a constant impedance $1/B$ in Fig. 10c. In all three cases, the reactive power $Q$ in (7) is varied proportionately to the real power $P$ (constant power factor). These bifurcation diagrams clearly indicate that the structure in the parameter space is the same for the three basic types, although the various bifurcations occur at different values. Specifically, the singularity is present for each special case, including a pure reactance for reactive load.

A. A Survey of the Dynamic Behavior Across Parameter Space and State Space

The main features of the bifurcation diagram of Figs. 3 and 4 are discussed here, but proofs will be deferred until

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Fig. 3. Qualitative representation of the bifurcation diagram.

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Fig. 4. Bifurcation diagram for the parameter set $T_2 = 5$, $T = 1.5$, $E_1 = 1.6$, $E_2 = 1.0$, $x_d = 1.2$, $x_q = 0.2$, $x = 0.1$, $Q_0 = 0.5 P$, $H = 0$, and $B = 0$.

---

Fig. 4'. Bifurcation diagram for the rudimentary system with SVC control for the parameters $T_2 = 10$, $T = 1.75$, $E_1 = 1.6$, $E_2 = 1.0$, $x_d = 1.0$, $x_q = 0.2$, $x = 0.1$, $Q_0 = 0.25 P$, $H = 0.12 P$ and $B = 0.12 P$.

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Section IV. Several different phenomena, represented by five bifurcation loci corresponding to three local bifurcations ($B_{SN}, B_{SH}, B_{ST}$) and two global bifurcations ($B_{SC}, B_{SC'}$) occur even for this rudimentary model. To a large extent, special points labeled $A$–$D$ in Fig. 3 determine the sequence of structural changes. These points give rise to five different levels of control gain indicated in
Fig. 3. Among them these levels produce all possible sequences of types of operation (typal regions \(1 \rightarrow 5\)) and intervening bifurcations as illustrated on the \(P-V\) curves of Fig. 5. Examples of state space behavior (phase portraits) for all eight typal regions are displayed in Figs. 6–8. Note that there are no stable equilibria for operating parameter values in typal regions \(3\) and \(5\). Moreover, the normal operating point \(x_i^4\) is stable only in regions \(1\), \(2\), \(4\), and \(6\) where \(1 = 1 \cup \bar{2}\) and \(2 = \bar{2} \cup \bar{7}\). The boundary \(E_{CD}\), shown as a dashed line in Figs. 3 and 4, is not a bifurcation since it only represents a change of \(x_i^4\) from a node \((\bar{1}, \bar{2})\) to a focus \((\bar{7}, \bar{2})\).

Within the region composed of \(1\), \(2\), \(4\), and \(6\), the position of \(x_i^4\) can be shifted by changing operating parameters within normal operating procedures (e.g., load increases) from any point to any other point without loss of stability. This region is called the feasibility region of the operating point here. Feasible operation at the high equilibrium ends at the boundary denoted by a heavy line in Fig. 3. This boundary, called the feasibility boundary, is composed here of segments of saddle node and Hopf bifurcation loci, joined at the point \(A\). The phase portraits, in particular the regions of stability, are illustrated in Figs. 6–8 in sequences along constant values of the control gain \(K\) at three levels (1, 2, and 5). Since between them, these three cover all types of operation and all transitions between the types of operation, this provides a complete set of illustrations. Our discussion will concentrate on the types encountered in sequence within level 2 (Fig. 7), because this gain level is most typical. But we will simultaneously comment on comparable events in levels 1 and 5. For the parameter values shown in Fig. 4, let us proceed from the left in Fig. 3 at the gain value \(K = 2.5\).

1) Type 1′ Behavior: (Fig. 7(a), with reference to the corresponding Figs. 6(a) for level 1 and 8(a) for level 5): At these light loads, there are two stable equilibria, \(x_i^1\) and \(x_i^2\), each to one side of the singular surface: \(x_i^3\) is a focus for parameter values above line \(E_{CD}\) in Fig. 3. There are three regions, two of which correspond to the regions of attraction of the stable equilibria (stability boundaries are shown by heavier dotted lines in Figs. 6–8). The region to the right of \(a - \psi - \gamma_h\) is the region of attraction of the stable equilibrium point \(x_i^1\) and the region to the left of \(a - \psi - \gamma_h\) is the region of attraction of the other stable equilibrium point \(x_i^2\). The third region contains no equilibria and all trajectories sink into the singular line, resulting in voltage collapse. The pivotal role is played by the point \(\psi\) on the load-flow manifold where \(E = 0\), but \(E_{ul} \neq 0\), which will be called a pseudo equilibrium. As will be shown in Section IV, this point \(\psi\) becomes an additional equilibrium point if the time scale is singularly transformed to eliminate the singularity. This already explains the special role of \(\psi\) as the limit point of infinitely many trajectories seen in some of the phase portraits or as anchor for stability boundaries in others like the one under consideration at the moment. There exist unique trajectories \(\gamma_i\) and \(\gamma_h\) on the left and right components of the load-flow manifold which converge to \(\psi\). These two trajectories, together with the singular line above \(\psi\) form the stability boundaries of the three regions.

All stability properties can be read off the phase portrait immediately. Upon clearing a disturbance, the system state may be initially anywhere in the state space. If the initial state is, for example, at point \(j\) [see Fig. 7(a)], then the system state will follow a trajectory with acceptable damping back to the stable equilibrium point at \(x_i^1\), the operating point of the system, following the time history in Fig. 9(a). Hence, initial state \(j\) has transient voltage stability in the traditional sense of power engineering. If the system should be at state \(k\) after the disturbance [Fig. 7(a)], then it will move on a trajectory to the low stable equilibrium point \(x_i^1\). The equilibrium \(x_i^1\) is of too low bus voltage for operation [Fig. 7(a)], but the system gets trapped there since \(x_i^1\) is stable. A system breakup by selective protection will follow. Industrial experiences of such entrapment have been reported [18]. However, as mentioned above, a valid proof of the stability of the components themselves in the full dynamic \((x, y)\) space does not exist because of modeling difficulties [11]. Finally, from an initial state at \(l\), the system will move to a voltage collapse at a singular point on the line \(\psi - a\) [Fig. 9(a)]. Note that with increasing gain \(K\) both equilibria gradually change from nodes to foci across line \(E_{CD}\). Also, note that in Fig. 8(b), the stability boundary \(\gamma_h\) bends back to a source point \(\bar{x}\) on the singular line, restricting the stability region of \(x_i^r\). This also portends the birth of a closed orbit in Fig. 8(c) at a higher load \(P\).

2) \(B_{SI}\), The Singularity Induced Bifurcation: (Figures 7(b) for level 2, 6(b) for level 1, and 8(f) on level 5): As the load increases to \((K, P) \in B_{SI}\), the lower stable equilibrium \(x_i^1\) crosses from the left to the right halfspace and changes into a saddle, i.e., a singularity induced bifurcation occurs. For parameter values on the bifurcation curve \(B_{SI}\), the equilibrium point lies on the singular line at the point \(\psi\) in the state space which coalesces with \(x_i\) here. All the trajectories in the left component \(LF_{low}\) converge to the singular line. The behavior of the stable equilibrium point \(x_i^1\) on the right component \(LF_{high}\) is not affected by the singularity induced bifurcation. It still is a stable focus with the region of attraction bounded by the curve \(a - x - \gamma_h\), where \(\gamma_h\) denotes the unique trajectory on the right halfspace which converges to the point \(x = \psi = x_i\). The trajectory \(\gamma_i\) loses its significance as stability boundary, but it still separates trajectories converging respectively to \(x_i\) or to the singular line. The same remarks apply to Fig. 6(b) except that the high equilibrium there is a node. In Fig. 8(f), the high equilibrium \(x_h \in B_{SI}\) (originally a focus) moves to the low side component \(LF_{low}\), resulting in the presence of a stable node \(x_i^1\) and a saddle \(x_h^1\) on the low side in Fig. 8(g) (region \(\bar{5}\)), while no equilibria exist on the high side there.

3) Type 2′ Behavior: (Figs. 7(c) and 6(c)): Upon increasing the load further, the lower equilibrium point \(x_i^1\) moves
into the right halfspace and becomes a saddle $x^*_1$. The high equilibrium point $x^*_1$ is still stable. Part of the stability boundary consists of the stable manifolds of the saddle $x^*_1$. The point $\psi$ itself becomes, respectively, a sink and a source for infinitely many trajectories on the left and right halfspaces. The only difference to note in Fig. 6(c) is that $x^*_1$ is a node.

No further changes occur in the phase portrait with increasing load in the left halfspace component (right halfspace in Fig. 8) and therefore only the right halfspace (left in Fig. 8) is described henceforth.

4) $B_{SC}$. The Saddle Connection Bifurcation and $B_{SC}$, the Singularity Connection Bifurcation: [Figs. 7(d) and 8(c)]: As the load increases, a saddle connection bifurcation occurs at the transition from region 7 into region 4. For values $(K, P) \in B_{SC}$, one each of the stable and unstable manifolds of the saddle $x^*_1$ coincide and form the homoclinic orbit $\Gamma$. This homoclinic orbit together with the equilibrium point $x^*_1$ forms the stability boundary for the stable focus $x^*_1$. Trajectories which lie inside converge to $x^*_1$ and the trajectories which lie outside, but which do not lie on the stable manifolds of the saddle $x^*_1$, converge to the singular line, i.e., experience voltage collapse. Because of the infinite "period" of the homoclinic orbit the motion of the state is very slow near $\Gamma$. This bifurcation is not encountered in levels 1 and 5 but the behavior at the singularity connection bifurcation $(K, P) \in B_{SC}$ in level 5 [Fig. 8(c)] where the trajectory $\Gamma$ of the stability boundary closes on itself at $\psi$ is quite similar to what happens at the homoclinic orbit. This orbit $\Gamma$ in Fig. 8(c) actually
becomes a homoclinic orbit in a transformed system where time has been rescaled (Section IV).

5) Type 4 and Type 6 Behavior: [Fig. 7(c) and 8(d)]: The structural changes upon moving from region (2) to region (4) through the saddle connection bifurcation ($B_{sc}$) in Figs. 7(c)–7(e) and, respectively, from region (7) to region (5) through the singularity connection bifurcation ($B_{sc}$) in Figs. 8(b)–8(d) are of a global nature. The local phase portraits near the two equilibria do not change at all: $x_i^*$ stays a saddle in Fig. 7(e) and the pseudo equilibrium $\psi$ also stays the same in Fig. 8(d); $x_i^*$ remains a stable focus in either case. The bifurcation occurs in the form of global rearrangements of the stable and unstable manifolds of $x_i^*$ at $B_{sc}$. They approach [region (2), Fig. 7(c)] and slide over each other forming a homoclinic orbit [Fig. 7(d)] where they coincide. Upon continuing parameter increase, they emerge in opposite relative position [Fig. 7(e)] while a limit cycle replaces the homoclinic orbit in region (4) in Fig. 7(e). Similarly the closed orbit in Fig. 8(e) where the origin and the end point of the stability boundary meet on the singularity at $\psi$ is replaced by a limit cycle in Fig. 8(d) upon load increase while $x_i^*$ and $x_i$ retain their character. Global structural changes of this type are considerably harder to analyze and detect than local bifurcations, but they have been intensively studied recently in connection with chaotic behavior of dynamical systems. The presence of homoclinic orbits even in this low-order model along with the fact that the dynamics is considerably more complex near the singularity strongly indicate the possible presence of chaotic behavior for certain parameter values in the general large power system. If so, such behavior would represent another form of voltage collapse regardless of detail.

As the load increases in region (4) or in region (5), the periodic orbit and hence the stability region for the high equilibrium $x_i^*$ shrink. Concurrently, the period of the limit cycle decreases from $\infty$ (for parameter values $(K, P) \in B_{sc}$ corresponding to the homoclinic orbit) to $\omega$ (for parameter values $(K, P) \in B_{sc}$ corresponding to a Hopf bifurcation with eigenvalues $\pm i\omega$). Fig. 9(b) shows time histories for points $v$ and $w$ which lie inside and outside the limit cycle, respectively, in Fig. 7(e).

6) $B_{pi}$, The Hopf Bifurcation and Type 5 or Type 7 Operation: [Fig. 7(f) and 7(e)]: The phase portrait for type
Fig. 8. Phase portraits for the control gain $K = 7.0$ in case 5 with other parameters as listed in Fig. 4.

Fig. 9. Time histories for the bus voltage $E$.

or operation given in Figs. 7(f) or 8(c) and for the Hopf bifurcation are qualitatively the same. The stability behavior of the high equilibrium point $x_h$ at the bifurcation value $B_H$ is already that of an unstable focus. Hence, except for the points on the stable manifold for $x^*_h$, all trajectories converge to a point on the singular line in the high component $L^F_{hi}$. Resulting in voltage collapse.

7) $B_{SN}$: The Saddle-Node Bifurcation: [Figs. 6(d), 7(g), and 8(h)]: For $(K, B) \in B_{SN}$, the well known saddle-node bifurcation occurs: the two equilibria merge at $x = x^*_h = x^*_s$. There are a variety of different combinations which occur at this point between the three gain levels 1, 2, and 5 studied. In level 2, which is our primary subject, the two merging equilibria are a saddle and an unstable node. The unstable focus in Fig. 7(f) becomes an unstable node (which is not a bifurcation) before reaching the saddle in Fig. 7(g). At the two other levels, a stable node meets a saddle, in the high component in case of level 1 [Fig. 6(d) following Fig. 6(c)] and in the lower half plane in case of level 5 [Fig. 8(h) following 8(g)]. In all cases a nonhyperbolic unstable equilibrium results at $x$.

8) Type 3 Behavior: [Figs. 6(e), 7(h), and 8(i)]: Finally, if $P$ is increased beyond its value on $B_{SN}$ the equilibrium disappears and all the trajectories on both the high and low sides converge to the singular line from where they also originated. Now every trajectory results in voltage collapse. This trivial situation occurs in region 3 and exemplary phase portraits are given in Figs. 6(e), 7(h), and 8(i).

B. Implications for the Operating Performance of the System

1) Security: The preceding analysis reveals a two-level cellular structure in the parameter space and the state space: The two governing features of the voltage dynamics problem are as follows: i) the region of attraction and its stability boundary in the state space and ii) the feasibility region and its feasibility boundary in the parameter space. Each stable equilibrium point has a stability region and its boundary in the state space, as shown by heavier dotted lines in Figs. 6–8. Each stable equilibrium has a feasibility region and its boundary in the parameter space, as shown for the high equilibrium by a heavier line in Fig. 3. A certain type of phase portrait in the state space persists for operating conditions within open regions of the parameter space. These regions here are called typical re-
ions (1) - (3). Each typical region has a typical phase portrait, that is, a transient behavior, displayed in Figs. 6-8 for each of the eight typical regions of the parameter space. Changes in the type of the phase portrait occur as parameters cross the bifurcation curves shown in Fig. 3. Some of the typical regions can be grouped together into a feasibility region where operation at a specific stable equilibrium point is feasible in the sense that the reference equilibrium can be shifted smoothly from any point in the region to any other point through normal operating practices, retaining its stable character. The boundary of the feasibility region is called feasibility boundary. The feasibility boundary for the high equilibrium point consists of a curve, \( B_{SN} - A - B_H \), connecting the lower half of the saddle-node bifurcation curve, \( B_{SN} \), with the Hopf bifurcation curve \( B_H \) at the codimension two bifurcation point \( A \) (Fig. 3). In the upper left-half plane (regions 1, 2, 3, and 6) operation is feasible at the high equilibrium point \( x_h \) which is stable there. For the low equilibrium point, \( x_l \), the feasibility boundary is actually wider at high gains than around \( x_h \). It consists of regions (1), (3), (7), and (5) bounded by the lower half of the singularity induced bifurcation, \( B_{SN} \), up to a point \( B \) and beyond point \( B \) by the upper half of the saddle-node bifurcation \( B_{SN} \). The lower equilibrium is unstable throughout this region but is typically not viable, mostly because of large voltage differences and heavy currents. Note that between them the stability boundaries of \( x_l \) and \( x_l \) consist of segments of saddle node, Hopf, and singularity induced bifurcations.

Following the clearing of a fault in the state space, all initial states which lie within the region of attraction of the equilibrium which serves as the operating point converge back to the equilibrium. So these form the region of transient stability. Stability boundaries are designated in each of Figs. 6-8 by heavier dotted lines. It should be observed that the region of attraction of \( x_l \), the usual operating point, gradually shrinks in size with increasing load and disappears at the feasibility boundary when one of the saddle node or Hopf bifurcations is crossed.

Equilibria are potential operating points provided they 1) are feasible, 2) have transient stability, 3) are viable (that is, their voltages and currents are within specified tolerances), and 4) are secure in the sense that they have feasibility, viability and transient stability with adequate margin for any fault contingency, i.e., any single fault which may occur on the system.

Figs. 4 7 are included to show that the results presented here also apply to other systems such as the system with SVC control instead of excitation control (8) and (9). It is important to note that the structure of the parameter space and the state space is qualitatively similar for either of the controls. To facilitate comparison, coordinated labeling is used between respectively Figs. 4 and 4', and Figs. 7(a) and 7'. Of course there is some warping, for instance the singular line \( a - a \) is not a straight line any more. Typical regions (4) and (5) do not appear in Fig. 4' and the singularity connection, the limit cycle and the Hopf bifurcation in \( B_{HC} \), regions (3) - (7) and \( B_H \), respectively, all appear in \( LF_{low} \) connected with the low equilibrium \( x_l \) instead of in \( LF_{high} \) as in the case in Fig. 4.

The results of this paper provide the proper basis for monitoring security. Viability is not discussed here, but it was discussed previously [19]. In the current state of art, viability is typically used as the sole basis for monitoring security.

2) Voltage Collapse Phenomena: Voltage collapse phenomena can be classified into dynamic (state space) and parametric (parameter space). Dynamic voltage collapse occurs when the post fault state lies outside the transient stability region and thus the high equilibrium is not regained. At least three subtypes exist: 1) The transient simply diverges out of practical range. 2) It converges at infinite speed to a singular surface where practical operation ends as in Fig. 9(a) for the trajectory from point \( I \) in Fig. 7(a). 3) It converges to the low equilibrium which is stable [Fig. 9(a), transient from point \( k \), of Fig. 7(a)]. Because of its stability the state is trapped at this impractical operating point and the system is then broken up by selective protection. Parametric voltage collapse occurs when the system (or its post fault composition) is located (as a result of load change or loss of transmission, generation etc.) in a region of the parameter space where it cannot return to a stable operating point. Two subtypes occur: i) a stable operating point does not exist (regions (3) and (5) for the example) or ii) a stable equilibrium does exist, but it is not practical for operation (such as the stable lower equilibrium in regions (7) and (8)).

3) Connections to the P-V (or Equivalently Q-V) Curve: In the approach of this paper, the analysis is based on the entire state space and the entire parameter space or its more active subspace (in this paper, three state variables and a two dimensional parameter subspace). For a general system both of these will be very large, but between them, they give all the data needed to study the whole system and give results which are well defined and meaningful. Traditionally, it is customary to attempt to use the "P-V curve" as the basis for analyzing even large and complex systems. The P-V (or Q-V) curve originated in the steady state analysis of a single transmission line and it is well suited for that [3]. It is unsuitable for analyzing the general large dynamic system or even the rudimentary system of this paper. It retains a single variable from the state space and from the parameter space—a woefully inadequate database. Researchers using this approach seem to put a major obstacle in their way even before they start. On the other hand, the P-V curve can be used for displaying the output of results in an effective way once such results are obtained by using the appropriate general analytical tools. This is illustrated in Fig. 5 of the paper for each of the eight typical regions of the rudimentary system. For instance, the results of level 1, Fig. 6 are illustrated on the P-V curve of Fig. 5(a). At these low gains, the upper branch is stable up to the saddle-node
bifurcation at the tip. The lower branch also has a stable (but nonviable) branch at loads up to the singularity induced bifurcation. No solution exists beyond the saddle node. As summarized in Figure 5(b), at level 2 stable operation ends on the upper branch at the Hopf bifurcation ahead of the saddle node. The Hopf bifurcation is preceded by a section where, though still stable, the region of attraction of \( x^h \) is restricted to the interior of an unstable limit cycle beyond \( B_{SC} \). On the lower branch, stability still ends at \( B_{SC} \). Finally Figure 5(c) summarises the results at level 5 and Fig. 8. Stability on the upper branch, that is for the high equilibrium, ends at the Hopf bifurcation, preceded by a section of shrinking region of attraction (starting at \( B_{SC} \) instead of \( B_{CC} \)). The lower branch or low equilibrium is stable everywhere up to the tip, \( B_{SC} \). The singularity induced bifurcation now occurs on the upper branch when the high equilibrium crosses the singularity into the low side. Note that the other two illustrations Figs. 5(c) and 5(d) display the sequence of events at levels 3 and 4 as also easily seen in Fig. 3.

This shows how the P-V curve, or equivalently, the Q-V curve can be useful as a tool for displaying results after they are obtained by proper analytical methods. It is however a serious drawback when used for analysis, as is common practice.

IV. BIFURCATION ANALYSIS

A. An Equivalent Smooth Dynamic System:

Eliminating \( E' \), \( E_c \), \( \delta \), \( \delta' \), and \( \delta'' \) from (3)-(7) yields

\[
E' = v(E, P) = \frac{1}{E} \sqrt{\chi^2 (P^2 + (qP)^2) + 2x'E^2 (qP) + E^4}
\]

(18)

\[
E_c = u(E, P) = \frac{1}{E} \sqrt{\chi^2 (P^2 + (qP)^2) + 2x'E^2 (qP) + E^4}.
\]

(19)

Hence, in the variables \( E \) and \( E_{fd} \) the dynamical equations now read

\[
\dot{E} = \left( -\frac{x + x_d}{T_{d0}x^2} v(E, P) + \frac{x_d - x_{d}'}{T_{d0}x^2} \frac{E^3}{v(E, P)} + \frac{E_{fd}}{T_{d0}} \right) \frac{1}{\dot{v}(E, P)}
\]

\[+ \frac{x_d - x_{d}'}{T_{d0}v(E, P)} \frac{qP}{v(E, P)} + \frac{E_{fd}}{T_{d0}} \right) \frac{1}{\dot{v}(E, P)} \]

\[= \left( -\frac{E_{fd}}{T} - \frac{K}{T} u(E, P) + \frac{E_{fd}^0 + E, K}{T} \right) \dot{v}(E, P).\]

(20)

Note that the same letters \( E \) and \( E_{fd} \) are used to denote the state variables in (20), (21), (23) and (24) even though the time scale has been changed from \( t \) in (20) and (21) to \( \tau \) in (23) and (24), according to the transformation

\[
\frac{dt}{d\tau} = \dot{v}(E(\tau), P).
\]

(25)

Also the dot denotes the respective time derivative in either \( t \) or \( \tau \). The phase portrait of the system \( \Sigma \) is exactly the same as that of the system given by (20) and (21) except for the following interpretational changes: i) The singular line no longer plays a special role. Points on it are now just ordinary points in the state space. ii) The time orientation of the trajectories is reversed to the left of the singular line. iii) The state \( E, E_{fd} \) moves along the same trajectories, but with speed

\[
\frac{dE}{d\tau} = \frac{dE}{dt} \dot{v}(E(\tau), P)
\]

and

\[
\frac{dE_{fd}}{d\tau} = \frac{dE_{fd}}{dt} \dot{v}(E(\tau), P).
\]

(26)
Away from the singularity, the collection of integral curves is identical for the systems given by (20) and (21) and for \( \Sigma \). In the system given by (20) and (21), the variable \( E \) approaches the singular line with infinite speed. For \( \Sigma \) this speed is finite and the speed of \( E_{\text{sing}} \) at the singular line becomes zero. Hence, at every point where \( \dot{E} \neq 0 \), the direction field is transversal to the singular line and there exists a unique trajectory passing through this point, connecting half trajectories on \( E < E_{\text{sing}} \) with corresponding half trajectories on \( E > E_{\text{sing}} \). Therefore, the phase portrait of \( \Sigma \) gives a canonical extension to the singular line of the phase portrait of the system with singularity given by (20) and (21). It follows from (23) that at \( E = E_{\text{sing}} \) the derivative \( \dot{E} \) vanishes at a unique point \( \psi \) given by

\[
E_{\text{fd}} = \frac{x + x_d}{x'} v(E_{\text{sing}}, P) \]

(27)

This point \( \psi \) becomes an isolated equilibrium point for the new system \( \Sigma \). Observe that for the original system, \( E_{\text{fd}} \) in (21) does not vanish at \( \psi \), so that \( \psi \) is not an equilibrium of the original system. This justifies naming \( \psi \) a pseudo equilibrium point.

**B. The Equilibrium Manifold:**

The static structure of equilibria away from the singular line will be analyzed next. It follows from (24) that when \( \dot{E}_{\text{fd}} = 0 \), then

\[
E_{\text{fd}} = E^0_{\text{fd}} + E, K - u(E, P) K
\]

(28)

and substituting this relation into (23) with \( \dot{E} = 0 \) gives

\[
E_{\text{fd}} = \frac{x + x_d}{x'} v(E, P) - \frac{x - x_d}{x'} \frac{E^2 + x'(qP)}{v(E, P)} - E^0_{\text{fd}} + K(u(E, P) - E) = 0.
\]

(29)

Since (28) gives an explicit formula for \( E_{\text{fd}} \) as function of \( (E, K, P) \), the set of equilibria can be visualized in \( (E, K, P) \) space. This set is denoted by \( EQ \) and called the equilibrium manifold with the understanding that the \( E_{\text{fd}} \) component is defined by (28). Away from the set \( \{(E, P); u(E, P) = E\} \), \( EQ \) is a smooth two-dimensional manifold. This can be seen by solving (29) for \( K \) as

\[
K = \frac{1}{E, -u(E, P)} \left( -E^0_{\text{fd}} + \frac{x + x_d}{x'} v(E, P) - \frac{x - x_d}{x'} E^2 + x'(qP) \right).
\]

(30)

The geometric shape of \( EQ \) will now be summarized in the following statements. The proofs require straightforward and elementary, though tedious manipulations, which will not be included here [21]. Set

\[
k_1(E, P) = \frac{1}{E, -u(E, P)} \]

\[
k_2(E, P) = \frac{x + x_d}{x'} v(E, P) - \frac{x - x_d}{x'} E^2 + x'(qP) - E^0_{\text{fd}}.
\]

(32)

1) Let \( P_3 \) be the (unique) positive root of the equation

\[
E^2_3 - 4 x(qP) \left[ E^2_3 - 4 x^2 P^2 \right] = 0.
\]

For \( P < P_3 \) the function \( k_1 \) has two poles \( E_{P, \pm} \) which are the positive roots of

\[
2 E^2_{P, \pm} = E^2_3 - 2 x(qP) \pm \sqrt{E^4_3 - 4 x(qP) E^2_3 - 4 x^2 P^2}.
\]

For \( P = P_3 \) a unique pole is given by

\[
2 E^2_{P, 0} = E^2_3 - 2 x(qP_3)
\]

and there are no poles for \( P > P_3 \). The function \( k_2 \) is strictly convex for all parameter values. There exists a unique parameter value \( P_2 < P_3 \) with the property that \( k_2 \) has two zeros \( E_{x, \pm} \) for \( P < P_2 \), a double zero for \( P = P_2 \), and no zeros for \( P > P_2 \). Furthermore, there exists a unique parameter value \( P_3 = P_3 \) where \( E_{P, \pm} \) and \( E_{x, \pm} \) cancel in a pole-zero cancellation (Fig. 11). 2) Let \( P \in (0, P_2) \). Then two load-flow solutions with positive \( K \) exist. For \( P = P_3 \), one of the two load-flow solutions is always given by \( E_{P, \pm} = E_{P, \pm} \) independently of \( K \). Additional load-flow solutions which represent control with positive feedback exist for negative \( K \) (as shown in Fig. 11). Some of these are stable, but they are associated with impractical voltage and current combinations.

3) Let \( P \in (P_2, P_3) \). Then the function \( K(\cdot, P) \) is positive on \( (E_{P, \pm}, E_{P, \pm}) \) and negative on \( (0, E_{P, \pm}) \) and \( (E_{P, \pm}, \infty) \). For all \( E \in (E_{P, \pm}, E_{P, \pm}) \) we have that \( \partial K / \partial E > 0 \). Furthermore, \( K \) is quasiconvex in \( E \) over the interval \( (E_{P, \pm}, E_{P, \pm}) \). In particular, in this interval \( K \) has a unique global minimum at the stationary point where \( \partial K / \partial E \) is positive, and there are two load-flow solutions for levels \( K < K_{\text{min}}(P) \), a unique one for \( K = K_{\text{min}}(P) \) and none for \( K < K_{\text{min}}(P) \). There exist no load-flow solutions with \( K \) positive for \( P > P_3 \).

Load-flow solutions corresponding to typical control gain values, \( K \), are shown in Fig. 11, which displays the warping of the \( K(E, P) \) curve with gradual increase of \( P \) from 0 to \( P_3 \) including the history of transitions at \( P_1, P_2, \) and \( P_3 \). They imply the shape of the equilibrium manifold \( EQ \) (Fig. 12).

**C. Bifurcation Loci:**

1) **Parameter Values:** \( P_2 < P < P_3 \): Here the structure of the equilibrium manifold will be analyzed further for parameter values \( P \) between \( P_2 \) and \( P_3 \). Let \( EQ_+ \) include \( \{(E, K, P) \in EQ : K > 0, P \in (P_2, P_3)\} \).
The geometric shape of $EQ_+$ (Fig. 12) follows from statements 1–3 above. For $P \in (P_1, P_2)$ fixed, the graph of $K$ is strictly quasiconvex and these graphs are increasing with $P$ for $E$ fixed. Let

$$SN = \{(E, K, P) \in EQ_+ : K'(E, P) = 0\}$$

(34)

denote the crest of the equilibrium manifold. (Here $K'$ again denotes the partial derivative with respect to $E$.) Notice that $SN$ divides $EQ_+$ into two halves. According to the value of the $E$-coordinate they will be called the high- and low-side respectively. Furthermore, since $(\partial K/\partial P)(E, P) > 0$, the equation $K'(E, P) = 0$ can always be solved for $P$ as a function of $E$. Since

$$dP = -\frac{K'(E, P)}{\partial K/E}(E, P)$$

(35)

it also follows that this function is increasing for points $(E, K, P)$ on the high side and decreasing for points on the low side. This establishes the structure of the popular $P-V$ curves as a slice of constant control gain $K$.

Next the bifurcation loci on the manifold $EQ_+$ will be established. The bifurcation diagram given in Fig. 3 is simply the projection of this set into the parameter space. First the loci where the Jacobian $DF$ of the system $\Sigma$ given by (23) and (24) has an eigenvalue 0 will be computed. A direct calculation shows that

$$\det(DF)|_{E \in EQ_+} = \frac{E_r - u(E, P)}{TT_{\partial 0}}K'(E, P)v'(E, P).$$

(36)

Since $E_s \neq u(E, P)$ for this parameter range, the only zeros occur for $SN$ and for

$$TC = \{(E, K, P) \in EQ_+ : v'(E, P) = 0\}. (37)$$

Each of these loci is the image under the map $K = K(E, P)$ of the graph of a smooth function $E = E(P)$ in $(E, P)$-space. For $TC$ an explicit expression of this function is given by (22) which also shows that this function is increasing in $P$. For $SN$ the implicit function theorem has to be involved. Since $K$ is strictly quasiconvex it has a strict minimum at the stationary point. Hence, $K''(E, P) > 0$ and so the equation $K''(E, P) = 0$ can be solved for $E$ as a smooth function of $P$. Verifying that $(\partial K'/\partial P)(E, P)$ is positive, it follows from

$$\frac{dE}{dP} = -\frac{\partial K'}{\partial P} - \frac{K''(E, P)}{K''(E, P)}$$

(38)

that the resulting function is decreasing in $P$. In particular, there exists a unique point of intersection $B$ of the curves $SN$ and $TC$.

For the two-dimensional system the Jacobian $DF$ has a double eigenvalue at zero or purely imaginary eigenvalues if and only if the trace of $DF$.

$$\text{tr}(DF) = \frac{\partial f_1}{\partial E} + \frac{\partial f_2}{\partial E_{\beta}}$$

(39)

vanishes. The equation $\text{tr}(DF) = 0$ is equivalent to a quartic polynomial equation in $E^2$ of the form

$$h(E, P) = -h_1E^4 - h_2E^3 + h_3E^2 + h_4E + h_0 = 0$$

(40)

where all the coefficients $h_i$ are positive. The $h_i$ can easily be computed [21]. The function $h$ is the product of $\text{tr}(DF)$ and a positive function. Since there is exactly one change of sign in these coefficients, it follows from Descartes' rule [22] that, for any given $P$, these equations have a unique positive real root which is simple. Since the polynomial is quartic in $E^2$, in principle this root can be calculated explicitly. Furthermore, because the root is simple, $h(E, P) = 0$ (and then in fact $h'(E, P) < 0$ because the coefficient $h_1 > 0$ and so the equation $h(E, P) = 0$ can be solved for $E$ as a smooth function $E = E(P)$.
is a smooth curve embedded in $E_{Q}$. It can also be verified that this function $E = E(P)$ is increasing in $P$ over the parameter range $P_2 < P < P_3$ by showing that $(\partial h / \partial E)(E, P)$ is positive at $h(E, P) = 0$ [21]. As a consequence, there also exists a unique intersection of the curve $\hat{H}$ with $SN$ and this point will be denoted by $A$. The curves $\hat{H}$ and $TC$ do not intersect. This can easily be seen by substituting the relation (22) which defines $TC$ into the polynomial $h$. The result is always positive.

A sketch of these curves on the equilibrium manifold $E_{Q}$ (together with a curve $SC$ which will be discussed later) is given in Fig. 12. Let $H$ be the portion of $\hat{H}$ which lies on the high side of $E_{Q}$. It will now be shown that the curves shown in Fig. 12 constitute the bifurcation set of the dynamical system $\Sigma$. The curves $SN, TC,$ and $H$ correspond to a saddle node, a transcritical and a Hopf bifurcation, respectively. In the $\Sigma$ model the transcritical bifurcation takes over the role of the singularity induced bifurcation described in Section III. All three are local codimension one bifurcations. The points $A$ and $B$ correspond to local codimension two bifurcations and the curve $SC$ indicates a global saddle connection bifurcation. The bifurcation diagram in Fig. 3 is the projection of these bifurcation loci into the parameter space. In Fig. 3 the corresponding projected curves are denoted $B_{SN}, B_{SIL}$ (for $TC$) and $B_{H}$ while the labeling of the points is the same. It is an immediate consequence of the geometric shape of the equilibrium manifold that the curves $B_{H}$ and $B_{SIL}$ are tangent to $B_{SN}$ at the points $A$ and $B$ of intersection. Notice that Fig. 12 gives more precise information about the bifurcation loci since it also identifies whether the bifurcations are on the low or on the high side. For instance, the curves $TC$ and $H$ do not intersect on $E_{Q}$, but in the projection into the parameter space an intersection point which was labeled $D$ in Fig. 3 arises. It is clear from Fig. 12 that $D$ has no special significance since the corresponding equilibria lie on different sides of $E_{Q}$. The local codimension one bifurcations happen to occur simultaneously at different equilibria. This can, of course, also be seen analytically, but the geometry of the equilibrium manifold and the bifurcation curves beautifully illustrate this point. Also, the popular $P - V$ curves are obtained as slices of $E_{Q}$ for constant $K$. The intersections of these $P - V$ curves with the various bifurcation curves on the surface $E_{Q}$ identify the occurrence of the various bifurcations on the $P - V$ curve as displayed for various $K$ values in Fig. 5. These in turn can be connected over a sequence of $P - V$ curves at fixed gain values $K$ to present bifurcation loci in the hybrid space composed of $P$ and $E$.

a) The Saddle-Node Bifurcation: The curve $SN$ corresponds to the crest of $E_{Q}$, i.e., for $P$ fixed the corresponding $K$-value is the minimum of the function $K(E, P)$ on $(E_{p_+}, E_{p_-})$. It is clear from the geometric shape of the equilibrium manifold that a saddle-node bifurcation occurs along $SN$. Also, if the control gain $K$ is kept constant and if $P$ is varied as parameter, then Sotomayor’s theorem (e.g., [23, theorem 3.4.1] or [24]) applies at points $(E, P, K) \in SN$ where the Jacobian $\partial F$ has 0 as simple eigenvalue and where a certain technical transversality condition is satisfied. Since the trace of $\partial F$ vanishes only at $A$ on $SN$, it follows that all points away from $A$ have 0 as simple eigenvalue (whereas 0 is a double eigenvalue of $\partial F$ at $A$). Furthermore, it can be verified that the transversality condition for the saddle-node bifurcation is only violated at the point $B$ [21]. Hence, in all points on $SN$ except for $A$ and $B$ a saddle-node bifurcation occurs.

b) The Transcritical Bifurcation: For fixed parameters $(K, P)$, the equation $v'(E, P) = 0$ also defines the singular line in the state space of the original model [see (10) to (12)]. Hence, for points in $TC$, one of the two equilibria has merged with the pseudo equilibrium $\psi$. The equilibrium $\psi$ gives rise to a second equilibrium manifold shown as $Ψ$ in Fig. 12. Since the location of $\psi$ is independent of $K$, $\psi$ simply is a vertical wall over the curve $v'(E, P) = 0$ in $(E, P)$-space. It intersects $E_{Q}$ in $TC$ and this curve corresponds to the singularity induced bifurcation described in Section III. If $K$ is kept fixed and $P$ is varied, then it is again evident from the geometric shape of the equilibrium manifold $E_{Q}$, that a transcritical bifurcation occurs at all points on TC except for $B$ (the two curves of equilibria are given by $E_{Q}\cap (K = constant)$ and the curve $v'(E, P) = 0$ lifted in the plane corresponding to this level of $K$). An analytical justification can be given by verifying the conditions of the transcritical bifurcation theorem [23, section III-D]. Like Sotomayor’s theorem for a saddle-node bifurcation, this theorem requires that $0$ is a simple eigenvalue of the Jacobian $\partial F$ and that a certain technical transversality condition holds. Since $\hat{H}$ and $TC$ do not intersect, $0$ is always a simple eigenvalue on $TC$ and it can be verified that, like for $SN$, the transversality condition is only violated at the point $B$. Hence, the local structure of the phase portraits (in state and parameter spaces) for points near $TC$ is known.

c) The Local Codimension Two Bifurcation at $B$—A Pitchfork Bifurcation: For the parameter values corresponding to the point $B$ a nongeneric codimension two bifurcation occurs as a result of a simultaneous occurrence of a saddle node and a transcritical bifurcation. (Note that both of these bifurcations occur for fixed $K$ when only the parameter $P$ is varied. Therefore, this is not a generic situation.) The two equilibria of $E_{Q}$ merge and disappear in a standard saddle-node bifurcation. The saddle-node bifurcation just happens to occur in the same location in the state space which is taken by $\psi$. The result is a so-called pitchfork bifurcation [23]. The typical pitchfork diagram arises as the intersection of $E_{Q}$ with the equilibrium manifold for $\psi$ in the level set $K = K_{B}$ corresponding to the value at $B$.

d) The Hopf Bifurcation: A Hopf bifurcation occurs when the Jacobian $\partial F$ has a simple pair of complex
eigenvalues which crosses the imaginary axis (while no other eigenvalues have zero real parts). These conditions hold on H. By construction \( \text{tr}(DF) = 0 \) on ̃H and H is the part of ̃H where \( \det(DF) > 0 \). Hence, DF has a simple pair of imaginary eigenvalues on H. Fix K and let P be the varying parameter. Recall that near H the intersection of the cross-section (K = constant) with the equilibrium manifold can then be described by a decreasing smooth function \( E = \tilde{E}(P) \). Furthermore, as was explained earlier, if \( (\tilde{E}(P) = K, P) \) ∈ H then \( \tilde{h}(\tilde{E}, P) < 0 \) and \( \tilde{f}/\tilde{P}(\tilde{E}, P) > 0 \) \[21\]. Therefore

\[
\frac{d}{dP} h(\tilde{E}(P), P) = h(\tilde{E}, P) \cdot \frac{d\tilde{E}}{dP} + \frac{\partial h}{\partial P}(\tilde{E}, P) > 0
\]

(42)

for the parameter value \( P_H \) corresponding to H. Since \( h(\tilde{E}, P) \) is a positive multiple of \( \text{tr}(DF) \), this implies that the eigenvalues of DF cross the axis from the left to the right, i.e., the required transversality condition is satisfied. Hence, the Hopf bifurcation theorem (e.g., \[23\], theorem 3.4.2) applies at every point on H. If a certain coefficient a does not vanish, this theorem gives the existence of a surface of periodic solutions which are stable if \( a < 0 \) and unstable if \( a > 0 \). In the appropriate coordinates (x, y) this surface of periodic solutions agrees up to second order with the paraboloid

\[
P - P_H = -\frac{a}{d}(x^2 + y^2) \quad \text{where} \quad d = \frac{d}{dP} \text{Re}(\lambda(P))|_{P = P_H}.
\]

(43)

Up to a positive multiple, d agrees with \( (d/dP) \) \( h(\tilde{E}(P), P) \) and hence is positive. It is well known how to calculate \( a \) (e.g., \[23\], page 152) and it can be verified (numerically) that \( a > 0 \). It follows that there exist unstable periodic orbits for \( P < P_H \) and no periodic orbits for \( P > P_H \) for P close to \( P_H \).

At the point \( A \) in the frontier of H the two purely imaginary eigenvalues merge at zero and DF has 0 as a double eigenvalue. The interplay of the Hopf bifurcation with the saddle-node bifurcation gives rise to a local codimension two bifurcation at A which has global consequences. Since both parameters, K and P, are relevant for this behavior, this is still a generic bifurcation.

e) The Local Codimension Two Bifurcation at A: The curves H and SN intersect in A. The local structure of the bifurcation set and the corresponding phase portraits of the system near \( A \) are well understood under certain generic assumptions which relate to the normal form of the dynamical system near the equilibrium \( (E_A, E_{idA}) \) \[23\], sections VII-B and VII-C). If these conditions are satisfied, then periodic orbits are generated (or destroyed) in a saddle connection bifurcation which occurs along a curve SC that originates at the point \( A \). This structure will be described in more detail and it will be indicated how these conditions can be verified for the system under consideration. Even for the two-dimensional system \( \Sigma \), in general it is not possible to give explicit formulas for the parameters \( K_A \) and \( P_A \) which define the point \( A \). However, once numerical values are assigned to the remaining constants, it is no problem to calculate \( A \) as the intersection of \( SN \) and \( ̃H \). For instance, if the parameters correspond to the values used in Figs. 6–8, then

\[
K_A = 1.8787, \quad P_A = 0.8783, \quad E_A = 0.7157, \quad E_{idA} = 1.023, \quad \text{and} \quad E_{idA} = 2.0012.
\]

The Taylor expansions of (10)–(12) at \( A \) take the form

\[
\dot{\tilde{E}}' = p_1 \tilde{P} + (a_{11} + p_2 \tilde{P}) \tilde{E} + a_{12} \tilde{E}_{id} + b_1 \tilde{E}^{2} + r_1 (44)
\]

\[
\tilde{E}_{id} = \left\{ c_1 \tilde{K} + p_3 \tilde{P} \right\} + (a_{21} + p_2 \tilde{P} + c_2 \tilde{K}) \tilde{E}' + b_2 \tilde{E}^{2} + a_{22} \tilde{E}_{id} + r_2 (45)
\]

where

\[
\tilde{E}' = E' - E_A, \quad \tilde{E}_{id} = E_{id} - E_{idA}, \quad \tilde{K} = K - K_A
\]

and \( r_1 \) and \( r_2 \) are higher order terms which are cubic in \( \tilde{E}, \tilde{E}_{id} \) and quadratic in \( K \) and \( P \). Setting

\[
z_1 = \tilde{E}' - \frac{a_2}{2a_1} \quad \text{and}
\]

\[
z_2 = a_{12} \tilde{E}_{id} + a_3 + (a_4 + a_{11}) \tilde{E}' + b_1 \tilde{E}^{2} (46)
\]

where

\[
a_1 = a_{12}^2 - a_{22} b_1, \quad a_2 = a_{12} \left\{ p_1 \tilde{P} + c_2 \tilde{K} \right\} + a_{11} p_2 \tilde{P}, \quad a_3 = p_1 \tilde{P}, \quad \text{and} \quad a_4 = p_2 \tilde{P}.
\]

The resulting system is of the form

\[
\dot{z}_1 = z_2 + r_3, \quad \dot{z}_2 = \mu_1 + \mu_2 z_3 + a_1 z_1^2 + 2b_1 z_1 z_2 + r_4
\]

(47)

with

\[
\mu_1 = a_{12} k_1 \tilde{K} + a_{12} p_3 \tilde{P} + a_{11} p_1 \tilde{P}, \quad \mu_2 = \left( a_4 - \frac{b_1}{a_1} a_2 \right)
\]

and \( r_3 \) and \( r_4 \) are higher order terms. Because of the normalization of the Jacobian DF at the equilibrium, it follows that

\[
\mu_1(K_A, P_A) = 0 \quad \text{and} \quad \mu_2(K_A, P_A) = 0 (48)
\]

so that the point \( A \) corresponds to \( (\mu_1, \mu_2) = (0, 0) \). The local structure of the bifurcation set near \( (\mu_1, \mu_2) = (0, 0) \) and the corresponding phase portraits near the equilibrium are described in \( [23, \text{section VII-C}] \). Using a singular blow up, the equations are transformed into a one-parameter family of systems which for parameter value 0 reduces to a Hamiltonian system that has a homoclinic orbit. Then applying Melnikov theory it can be shown that the homoclinic orbit persists for small positive parameter values (see \[23\], sections IV-F and VII-C). Briefly, it is
shown in [23, section VII-C] that the semiparabola

$$\bar{SC}: \mu_1 = - \frac{49}{25\beta^2} \mu_2^2, \quad \mu_2 \geq 0$$

(49)

(where $\beta = (2b_1/a_1)$ is a quadratic approximation to a true bifurcation curve $SC$ at $(\mu_1, \mu_2) = (0, 0)$ where the system undergoes a saddle connection bifurcation. The saddle-node bifurcation curve is given by $\mu_1 = 0, \mu_2 \neq 0$ and the Hopf bifurcation occurs along

$$\hat{H}: \mu_1 = - \frac{1}{\beta^2} \mu_2^2, \quad \mu_2 \geq 0.$$  

(50)

Let $G$ be the domain between the curves $\hat{H}$ and $\bar{SC}$. If a parameter value $\mu_1 < 0$ near zero is fixed, then it follows from the Hopf bifurcation theorem that unstable periodic orbits are born at $\hat{H}$ whose initial period is given by $\omega$ if the eigenvalues of the Jacobian are $\pm i\omega$. Using Melnikov theory again it can be verified that the system has a unique unstable limit cycle for each pair of parameters $(\mu_1, \mu_2) \in G$ near $(0, 0)$. Furthermore, as $\mu_2$ decreases from $\hat{H}$ to $\bar{SC}$ for $\mu_1 < 0$ fixed, the periods of these closed orbits become unbounded. The homoclinic orbit which occurs for $(\mu_1, \mu_2) \in SC$ lies in the closure of this family of periodic orbits. There exist no periodic orbits for $(\mu_1, \mu_2)$ below and close to $SC$ [23, theorem 6.1.1]. In a certain way, this result gives the counterpart to the Hopf bifurcation theorem for saddle connection bifurcations.

These results completely describe the bifurcation set and the qualitative structure of the phase portrait of the system $\Sigma$ for parameter values near $A$. It only needs to be verified that $a_1 \neq 0, b_1 \neq 0$. For instance, for the numerical values used above, the coefficients are given as $a_1 = 5.46, b_1 = -6.076$ and so the nonzero conditions are satisfied. Moreover, the nonzero conditions on the coefficients $a_1$ and $b_1$ are satisfied for a reasonable range of operating conditions. The local structure of the bifurcation diagram in Fig. 3 near the point $A$ is therefore established. In Fig. 12 the curve $B_{SC}^1$ (near $A$) has been lifted to a curve $SC$ on the low side of $EQ_1$. This, of course, is arbitrary, but it seems reasonable since the homoclinic orbit is homclonic to the low equilibrium.

f) Global Properties: The discussion given under e) establishes the existence of a curve $SC$ near $A$ with the property that for $(E, K, P) \in SC$ there exists an orbit homoclinic to $x_i$. This analysis is local and not valid far away from $A$. Numerical calculations show that the saddle connection bifurcation persists. For a fixed value of $K$, unstable periodic orbits are born around the high equilibrium in the Hopf bifurcation (Figs. 3, 4, and 4'). For decreasing $P$, the periods increase and become unbounded for $P$ approaching a value $P_{SC}$ where they change into a homoclinic orbit. For values of $K$ in level 2, this orbit is homoclinic to the low equilibrium (saddle connection), for values of $K$ in higher levels it is homoclinic to the pseudo equilibrium $\psi$ (singularity connection). The switch occurs at the point $C$ where the low equilibrium and the pseudo equilibrium merge. Furthermore, for $P \in (P_{SC}, P_{P1})$ there exists a unique unstable limit cycle and no periodic orbits exist for $P < P_{SC}$. Hence, the same behavior which holds near $A$ is valid for all levels $K$. Presently, these statements are only based on numerical evidence, not on a mathematical analysis. Also, global aspects of the phase portrait have not been established yet. But our numerical calculations are sufficient to establish the dynamical behavior of the system $\Sigma$ for the power system over the practically relevant regions of the state and parameter space.

2) Parameter Values $0 < P < P_2$: In Section IV-C-1, the analysis was concentrated on load values $P_2 < P < P_3$ where a large variety of events occur. It remains now to complete the analysis for $0 < P < P_2$. Let us go back to the equilibrium structure of the system for all power values $P$ discussed in Section IV-B as summarized in Fig. 11. From the geometric shape of the equilibrium surface and by Sotomayors' theorem, it can be seen that the saddle-node bifurcation occurs at the local minima of the equilibrium surface $EQ$ in Fig. 11. Note that the load $P = P_2$ has been defined as the load value when this minimum occurs for $K = 0$, i.e., the load which corresponds to a double zero for the $EQ$ surface in Fig. 11. Then the structure of the equilibrium surface directly implies that the surface $EQ$ has no minima for positive control $K$ gain values (i.e., negative feedback) when $P < P_2$. Hence, the saddle-node bifurcation does not occur for positive control gain values for all low loads ($P < P_2$). For $P_1 < P < P_2$ the saddle node occurs at negative control gain values, and below $P = P_1$, there is no saddle-node bifurcation (as there is no minimum for the surface $EQ$). So for positive control gain values, which correspond to the practically interesting case, the system always has two equilibria for low loads, i.e., when $0 < P < P_2$. Let us restrict the stability analysis which follows next to positive control gain values $K > 0$.

As shown in the previous subsections, the high equilibrium point $x_h$ is stable up to the saddle node or the Hopf bifurcations, whichever occurs first. The saddle node does not occur in this range as discussed above. The Hopf bifurcation disappears at the codimension two bifurcation, point $A$. For typical system parameter values, the load at point $A$, $P_i$ is higher than the load $P_2$ [21]. Hence, Hopf bifurcations do not occur at these low load values. So the high equilibrium point is stable for all practical control gains for low loads ($P < P_2$).

For the low equilibrium point $x_i$, the feasibility boundary consists of pieces of saddle node and singularity induced bifurcations. Again for the equilibrium point $x_i$, the saddle node never occurs in this parameter range. The transcritical (or singularity induced) bifurcation occurs under generic conditions at the points in the zero set $TC$ defined in (37). Define $P_{TC}$ as the load when the transcritical bifurcation occurs at $K = 0$. Then under light load conditions, i.e., for $P < P_{TC}$ and $P < P_2$, the low equilibrium is also stable for all practical control gain values.
V. CONCLUSION

In this paper, the voltage dynamics of a rudimentary, but representative power system has been analyzed. The simplicity and low dimension of the model allows us to completely analyze the structural features over important regions in state and parameter space. Though it cannot be expected that a similar comprehensive analysis can be achieved for the large power system in general, the results presented here still encourage research to establish the fundamental structures and important characteristics of both state and parameter spaces for the general system. Moreover, concepts introduced here, such as the feasibility region of a stable operating point and the geometrical boundary are easily extended to the general system, and the mathematical theory can be as in [11]. Also, the structural decomposition of the stability boundary of a stable equilibrium point into singular segments and analytic systems (consisting of trajectories) with hyperbolic and singular anchor points can be analyzed for a general nonlinear system of constrained ordinary differential equations under Morse–Smale like assumptions about the structure of its equilibria, periodic orbits, and intersections of stable and unstable manifolds.

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REFERENCES


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