Homework 10 Solution

Problems 1 and 2 are similar to the solutions of the first two problems (except for numbers)

Problem 1: change ε_r to 4 instead of 9 and coefficient of $\cos\theta$ to 2 instead of 3.

Problem 4.45 A 2-cm conducting sphere is embedded in a charge-free dielectric medium with $\varepsilon_{2r} = 9$. If $\mathbf{E}_2 = \hat{\mathbf{R}}3\cos\theta - \hat{\boldsymbol{\theta}}3\sin\theta$ (V/m) in the surrounding region, find the charge density on the sphere's surface.

Solution: According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s.$$

In the present case, $\hat{\mathbf{n}}_2 = \hat{\mathbf{R}}$ and $\mathbf{D}_1 = 0$. Hence,

$$\begin{aligned} \rho_s = {} & \hat{\mathbf{R}} \cdot \mathbf{D}_2|_{r=2\text{ cm}} \\ = {} & \hat{\mathbf{R}} \cdot \varepsilon_2(\hat{\mathbf{R}}3\cos\theta \quad \hat{\boldsymbol{\theta}}3\sin\theta) \\ = {} & 27\varepsilon_0 \cos\theta \quad (\text{C/m}^2). \end{aligned}$$

Problem 2: Change 150 change to 500

Problem 4.46 If $\mathbf{E} = \hat{\mathbf{R}}150$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

Solution: From Table 4-3, $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. $\mathbf{E}_2$ inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1R} = \rho_s, \qquad E_{1R} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

$$Q = E_R S\varepsilon_0 = (150)4\pi(0.05)^2\varepsilon_0 = \frac{3\pi\varepsilon_0}{2} \quad (\text{C}).$$

Problem 3

Problem 4.53 Use the result of Problem 4.52 to determine the capacitance for each of the following configurations:

(a) conducting plates are on top and bottom faces of rectangular structure in Fig. 4-35(a) (P4.53(a)),

(b) conducting plates are on front and back faces of structure in Fig. 4-35(a) (P4.53(a)),

(c) conducting plates are on top and bottom faces of the cylindrical structure in Fig. 4-35(b) (P4.53(b)).

Solution:

(a) The two capacitors share the same voltage; hence they are in parallel.

$$C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(5 \times 1) \times 10^{-4}}{2 \times 10^{-2}} = 5\varepsilon_0 \times 10^{-2},$$

$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(5 \times 3) \times 10^{-4}}{2 \times 10^{-2}} = 30\varepsilon_0 \times 10^{-2},$$

$$C = C_1 + C_2 = (5\varepsilon_0 + 30\varepsilon_0) \times 10^{-2} = 0.35\varepsilon_0 = 3.1 \times 10^{-12} \text{ F}.$$

(b)

$$C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(2 \times 1) \times 10^{-4}}{5 \times 10^{-2}} = 0.8\,\varepsilon_0 \times 10^{-2},$$

$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(3 \times 2) \times 10^{-4}}{5 \times 10^{-2}} = \frac{24}{5}\varepsilon_0 \times 10^{-2},$$

$$C = C_1 + C_2 = 0.5 \times 10^{-12} \text{ F}.$$

(c)

$$C_1 = \varepsilon_1 \frac{A_1}{d} = 8\varepsilon_0 \frac{(\pi r_1^2)}{2 \times 10^{-2}} = \frac{4\pi\varepsilon_0}{10^{-2}} (2 \times 10^{-3})^2 = 0.04 \times 10^{-12} \text{ F},$$

$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(\pi(r_2^2 - r_1^2)}{2 \times 10^{-2}} = \frac{2\pi\varepsilon_0}{10^{-2}} [(4 \times 10^{-3})^2 - (2 \times 10^{-3})^2] = 0.06 \times 10^{-12} \text{ F},$$

$$C_3 = \varepsilon_3 \frac{A_3}{d} = 2\varepsilon_0 \frac{(\pi(r_3^2 - r_2^2)}{2 \times 10^{-2}} = \frac{\pi\varepsilon_0}{10^{-2}} [(8 \times 10^{-3})^2 - (4 \times 10^{-3})^2] = 0.12 \times 10^{-12} \text{ F},$$

$$C = C_1 + C_2 + C_3 = 0.22 \times 10^{-12} \text{ F}.$$

Problem 4

Problem 4.55 Use the expressions given in Problem 4.54 to determine the capacitance for the configurations in Fig. 4.35(a) (P4.55) when the conducting plates are placed on the right and left faces of the structure.

Solution:

$$C_1 = \varepsilon_1 \frac{A}{d_1} = 2\varepsilon_0 \frac{(2 \times 5) \times 10^{-4}}{1 \times 10^{-2}} = 20\varepsilon_0 \times 10^{-2} = 1.77 \times 10^{-12} \text{ F},$$

$$C_2 = \varepsilon_2 \frac{A}{d_2} = 4\varepsilon_0 \frac{(2 \times 5) \times 10^{-4}}{3 \times 10^{-2}} = 1.18 \times 10^{-12} \text{ F},$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{1.77 \times 1.18}{1.77 + 1.18} \times 10^{-12} = 0.71 \times 10^{-12} \text{ F}.$$

Problem 5:

Problem 4.56 With reference to Fig. 4-37 (P4.56), charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent), and

(b) then find the electric potential and electric field at an arbitrary point $P(0, y, z)$.

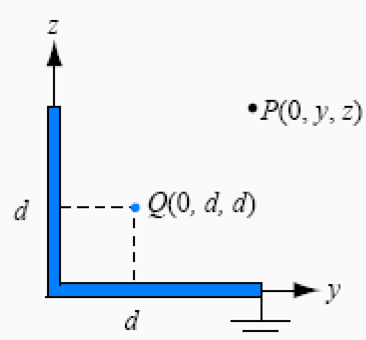

Figure P4.56: Charge Q next to two perpendicular, grounded, conducting half planes.

Solution:

(a) The original charge has magnitude and polarity $+Q$ at location $(0, d, d)$. Since the negative y-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location

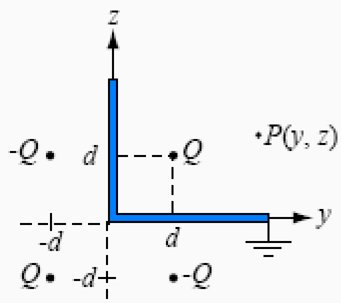

Figure P4.56: (a) Image charges.

$(0, d, -d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction. This ground plane in the x-z plane gives rise to the image charges of $-Q$ at $(0, -d, d)$ and $+Q$ at $(0, -d, -d)$.

(b) Using Eq. (4.47) with $N = 4$,

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z-d)|} - \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z-d)|} + \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d) + \hat{\mathbf{z}}(z+d)|} - \frac{1}{|\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d) + \hat{\mathbf{z}}(z+d)|}\right)$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2 \quad (y \quad d)^2 \quad (z \quad d)^2}} \quad \frac{1}{\sqrt{x^2 \quad (y \quad d)^2 \quad (z \quad d)^2}} \quad \frac{1}{\sqrt{x^2 \quad (y \quad d)^2 \quad (z \quad d)^2}} \quad \frac{1}{\sqrt{x^2 \quad (y \quad d)^2 \quad (z \quad d)^2}}\right)$$

$$- \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2 \quad y^2 \quad 2yd \quad z^2 \quad 2zd \quad 2d^2}} \quad \frac{1}{\sqrt{x^2 \quad y^2 \quad 2yd \quad z^2 \quad 2zd \quad 2d^2}} \quad \frac{1}{\sqrt{x^2 \quad y^2 \quad 2yd \quad z^2 \quad 2zd \quad 2d^2}} \quad \frac{1}{\sqrt{x^2 \quad y^2 \quad 2yd \quad z^2 \quad 2zd \quad 2d^2}}\right) \quad \text{(V)}.$$

From Eq. (4.51),

$$
\begin{aligned}
\mathbf{E} &= -\nabla V \\
&= \frac{Q}{4\pi\varepsilon}\left(-\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}} + \nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right. \\
&\qquad \left. -\nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}} + \nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right) \\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y-d)^2+(z-d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y+d)^2+(z-d)^2)^{3/2}}\right. \\
&\qquad \left. + \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y+d)^2+(z+d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y-d)^2+(z+d)^2)^{3/2}}\right) \quad \text{(V/m)}.
\end{aligned}
$$

Problem 6:

Problem 4.58 Use the image method to find the capacitance per unit length of an infinitely long conducting cylinder of radius a situated at a distance d from a parallel conducting plane, as shown in Fig. 4-39 (P4.58).

Solution: Let us distribute charge ρ_l (C/m) on the conducting cylinder. Its image cylinder at $z = -d$ will have charge density $-\rho_l$.

For the line at $z = d$, the electric field at any point z (at a distance of $d \quad z$ from the center of the cylinder) is, from Eq. (4.33),

$$\mathbf{E}_1 = -\hat{\mathbf{z}} \frac{\rho_l}{2\pi\varepsilon_0(d \quad z)}$$

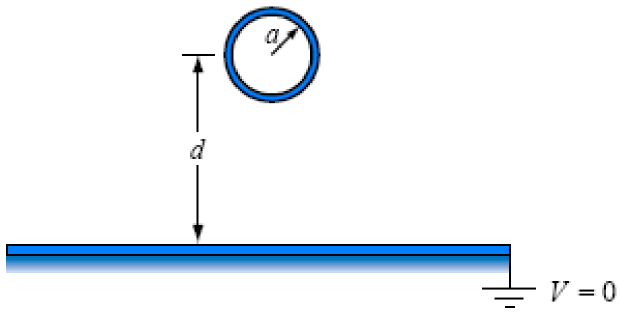

Figure P4.58: Conducting cylinder above a conducting plane (Problem 4.58).

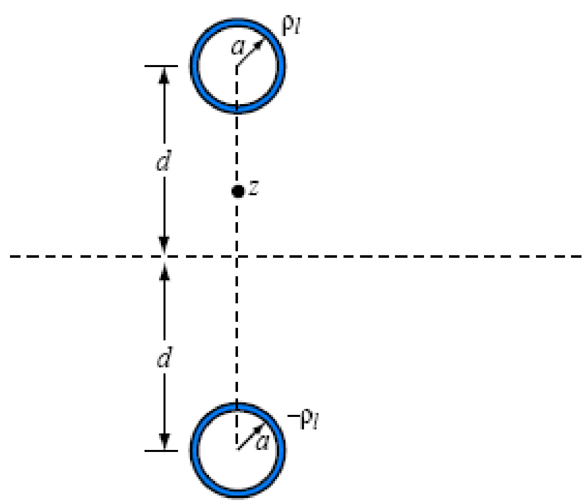

Figure P4.58: (a) Cylinder and its image.

where $-\hat{\mathbf{z}}$ is the direction away from the cylinder. Similarly for the image cylinder at distance $(d+z)$ and carrying charge $-\rho_l$,

$$\mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{(-\rho_l)}{2\pi\varepsilon_0(d+z)} = -\hat{\mathbf{z}}\,\frac{\rho_l}{2\pi\varepsilon_0(d+z)}\,.$$

The potential difference between the cylinders is obtained by integrating the total electric field from $z=-(d-a)$ to $z=(d-a)$:

$$\begin{aligned}
V &= -\int_2^1 (\mathbf{E}_1+\mathbf{E}_2)\cdot\hat{\mathbf{z}}\,dz \\
&= \quad \int_{-(d-a)}^{d\ \ a} \quad \hat{\mathbf{z}}\,\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{d\ \ z}\quad\frac{1}{d\ \ z}\right)\cdot\hat{\mathbf{z}}\,dz \\
&- \frac{\rho_l}{2\pi\varepsilon_0}\int_{(d\ \ a)}^{d-a}\left(\frac{1}{d\ \ z}+\frac{1}{d\ \ z}\right)dz \\
&= \frac{\rho_l}{2\pi\varepsilon_0}\left[\quad\ln(d\ \ z)\quad\ln(d\ \ z)\right]_{(d\ \ a)}^{d-a} \\
&= \frac{\rho_l}{2\pi\varepsilon_0}\left[\quad\ln(a)\ .\ \ln(2d\ \ a)\ .\ \ln(2d\ \ a)\quad\ln(a)\right] \\
&= \frac{\rho_l}{\pi\varepsilon_0}\ln\left(\frac{2d\ \ a}{a}\right).
\end{aligned}$$

For a length L, $Q=\rho_l L$ and

$$C = \frac{Q}{V} = \frac{\rho_l L}{(\rho_l/\pi\varepsilon_0)\ln[(2d-a)/a]},$$

and the capacitance per unit length is

$$C' = \frac{C}{L} = \frac{\pi\varepsilon_0}{\ln[(2d/a)\quad 1]} \quad \text{(C/m)}.$$