

Homework 11 solutions

Problem 1: 5.6

Problem 5.6 A 20-turn rectangular coil with side $l = 20$ cm and $w = 10$ cm is placed in the y - z plane as shown in Fig. 5-34 (P5.6).

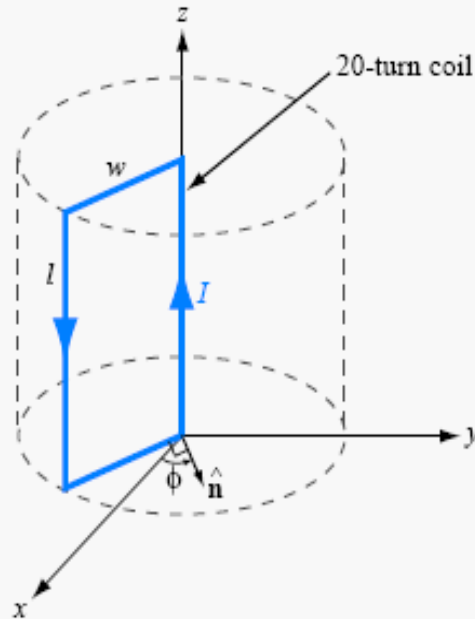


Figure P5.6: Rectangular loop of Problem 5.6.

- (a) If the coil, which carries a current $I = 10$ A, is in the presence of a magnetic flux density

$$\mathbf{B} = 2 \times 10^{-2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}2) \quad (\text{T}),$$

determine the torque acting on the coil.

- (b) At what angle ϕ is the torque zero?
 (c) At what angle ϕ is the torque maximum? Determine its value.

Solution:

- (a) The magnetic field is in direction $(\hat{\mathbf{x}} + \hat{\mathbf{y}}2)$, which makes an angle $\phi_0 = \tan^{-1} \frac{2}{1} = 63.43^\circ$.

The magnetic moment of the loop is

$$\mathbf{m} = \hat{\mathbf{n}}NIA = \hat{\mathbf{n}}20 \times 10 \times (30 \times 10) \times 10^{-4} = \hat{\mathbf{n}}6 \quad (\text{A}\cdot\text{m}^2),$$

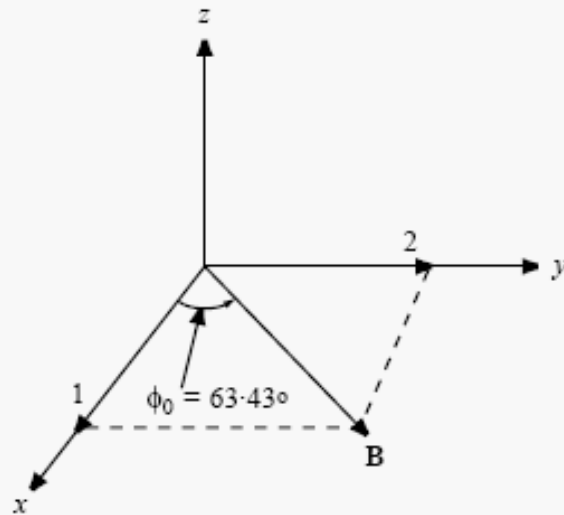


Figure P5.6: (a) Direction of \mathbf{B} .

where $\hat{\mathbf{n}}$ is the surface normal in accordance with the right-hand rule. When the loop is in the negative- y of the y - z plane, $\hat{\mathbf{n}}$ is equal to $\hat{\mathbf{x}}$, but when the plane of the loop is moved to an angle ϕ , $\hat{\mathbf{n}}$ becomes

$$\begin{aligned}\hat{\mathbf{n}} &= \hat{\mathbf{x}} \cos \phi, \hat{\mathbf{y}} \sin \phi, \\ \mathbf{T} = \mathbf{m} \times \mathbf{B} &= \hat{\mathbf{n}} 6 \times 2 \times 10^{-2} (\hat{\mathbf{x}} - \hat{\mathbf{y}} 2) \\ &= (\hat{\mathbf{x}} \cos \phi, \hat{\mathbf{y}} \sin \phi) 6 \times 2 \times 10^{-2} (\hat{\mathbf{x}} - \hat{\mathbf{y}} 2) \\ &= \hat{\mathbf{z}} 0.12 [2 \cos \phi - \sin \phi] \text{ (N}\cdot\text{m)}.\end{aligned}$$

(b) The torque is zero when

$$2 \cos \phi - \sin \phi = 0,$$

or

$$\tan \phi = 2, \quad \phi = 63.43^\circ \text{ or } 116.57^\circ.$$

Thus, when $\hat{\mathbf{n}}$ is parallel to \mathbf{B} , $\mathbf{T} = 0$.

(c) The torque is a maximum when $\hat{\mathbf{n}}$ is perpendicular to \mathbf{B} , which occurs at

$$\phi = 63.43 \pm 90^\circ = -26.57^\circ \text{ or } +153.43^\circ.$$

Mathematically, we can obtain the same result by taking the derivative of \mathbf{T} and equating it to zero to find the values of ϕ at which $|\mathbf{T}|$ is a maximum. Thus,

$$\frac{\partial T}{\partial \phi} = \frac{\partial}{\partial \phi} (0.12(2 \cos \phi - \sin \phi)) = 0$$

or

$$2 \sin \phi \cdot \cos \phi = 0,$$

which gives $\tan \phi = \frac{1}{2}$, or

$$\phi = -26.57^\circ \text{ or } 153.43^\circ,$$

at which $\mathbf{T} = \hat{\mathbf{z}}0.27 \text{ (N}\cdot\text{m)}$.

Problem 2: 5.12

Problem 5.12 Two infinitely long, parallel wires carry 6-A currents in opposite directions. Determine the magnetic flux density at point P in Fig. 5-38 (P5.12).

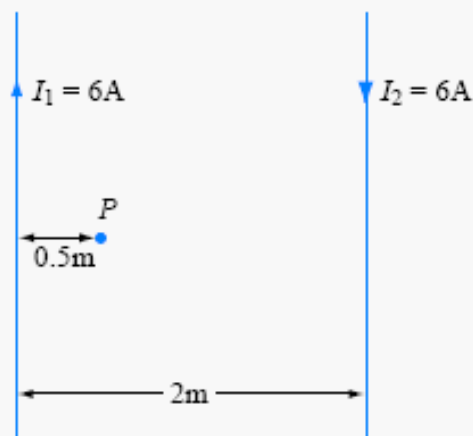


Figure P5.12: Arrangement for Problem 5.12.

Solution:

$$\mathbf{B} = \hat{\phi} \frac{\mu_0 I_1}{2\pi(0.5)} + \hat{\phi} \frac{\mu_0 I_2}{2\pi(1.5)} = \hat{\phi} \frac{\mu_0}{\pi} (6 + 2) = \hat{\phi} \frac{8\mu_0}{\pi} \quad (\text{T}).$$

Problem:3:5.14

Problem 5.14 Two parallel, circular loops carrying a current of 40 A each are arranged as shown in Fig. 5-39 (P5.14). The first loop is situated in the x - y plane with its center at the origin and the second loop's center is at $z = 2$ m. If the two loops have the same radius $a = 3$ m, determine the magnetic field at:

- (a) $z = 0$,
- (b) $z = 1$ m,
- (c) $z = 2$ m.

Solution: The magnetic field due to a circular loop is given by (5.34) for a loop in the x - y plane carrying a current I in the $+\hat{\phi}$ -direction. Considering that the bottom loop in Fig. P5.14 is in the x - y plane, but the current direction is along $-\hat{\phi}$,

$$\mathbf{H}_1 = \hat{z} \frac{Ia^2}{2(a^2 + z^2)^{3/2}},$$

where z is the observation point along the z -axis. For the second loop, which is at a height of 2 m, we can use the same expression but z should be replaced with $(z - 2)$. Hence,

$$\mathbf{H}_2 = \hat{z} \frac{Ia^2}{2[a^2 + (z - 2)^2]^{3/2}}.$$

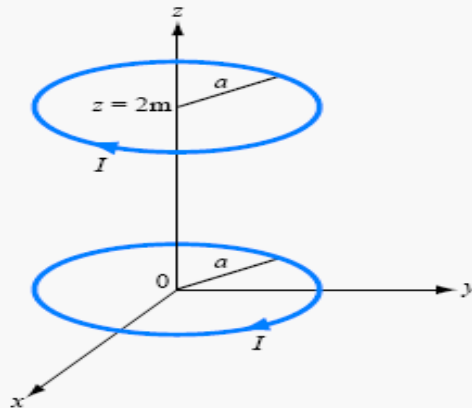


Figure P5.14: Parallel circular loops of Problem 5.14.

The total field is

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -\hat{z} \frac{Ia^2}{2} \left[\frac{1}{(a^2 + z^2)^{3/2}} + \frac{1}{[a^2 + (z - 2)^2]^{3/2}} \right] \text{ A/m.}$$

- (a) At $z = 0$, and with $a = 3$ m and $I = 40$ A,

$$\mathbf{H} = -\hat{z} \frac{40 \times 9}{2} \left[\frac{1}{3^3} + \frac{1}{(9 + 4)^{3/2}} \right] = -\hat{z} 10.5 \text{ A/m.}$$

- (b) At $z = 1$ m (midway between the loops):

$$\mathbf{H} = -\hat{z} \frac{40 \times 9}{2} \left[\frac{1}{(9 + 1)^{3/2}} + \frac{1}{(9 + 1)^{3/2}} \right] = -\hat{z} 11.38 \text{ A/m.}$$

- (c) At $z = 2$ m, \mathbf{H} should be the same as at $z = 0$. Thus,

$$\mathbf{H} = -\hat{z} 10.5 \text{ A/m.}$$

Problem 4: 5.16

Problem 5.16 In the arrangement shown in Fig. 5-41 (P5.16), each of the two long, parallel conductors carries a current I , is supported by 8-cm-long strings, and has a mass per unit length of 1.2 g/cm. Due to the repulsive force acting on the conductors, the angle θ between the supporting strings is 10° . Determine the magnitude of I and the relative directions of the currents in the two conductors.

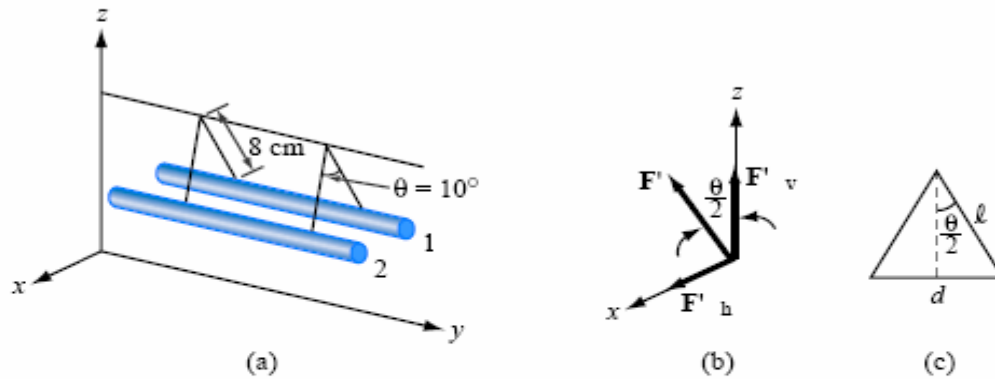


Figure P5.16: Parallel conductors supported by strings (Problem 5.16).

Solution: While the vertical component of the tension in the strings is counteracting the force of gravity on the wires, the horizontal component of the tension in the strings is counteracting the magnetic force, which is pushing the wires apart. According to Section 5-3, the magnetic force is repulsive when the currents are in opposite directions.

Figure P5.16(b) shows forces on wire 1 of part (a). The quantity F' is the tension force per unit length of wire due to the mass per unit length $m' = 1.2 \text{ g/cm} = 0.12$

kg/m. The vertical component of F' balances out the gravitational force,

$$F'_v = m'g, \quad (19)$$

where $g = 9.8 \text{ (m/s}^2\text{)}$. But

$$F'_v = F' \cos(\theta/2). \quad (20)$$

Hence,

$$F' = \frac{m'g}{\cos(\theta/2)}. \quad (21)$$

The horizontal component of F' must be equal to the repulsion magnitude force given by Eq. (5.42):

$$F'_h = \frac{\mu_0 I^2}{2\pi d} = \frac{\mu_0 I^2}{2\pi[2\ell \sin(\theta/2)]}, \quad (22)$$

where d is the spacing between the wires and ℓ is the length of the string, as shown in Fig. P5.16(c). From Fig. 5.16(b),

$$F_h' = F' \sin(\theta/2) = \frac{m'g}{\cos(\theta/2)} \sin(\theta/2) = m'g \tan(\theta/2). \quad (23)$$

Equating Eqs. (22) and (23) and then solving for I , we have

$$I = \sin(\theta/2) \sqrt{\frac{4\pi\ell m'g}{\mu_0 \cos(\theta/2)}} = \sin 5^\circ \sqrt{\frac{4\pi \times 0.08 \times 0.12 \times 9.8}{4\pi \times 10^{-7} \cos 5^\circ}} = 84.8 \quad (\text{A}).$$

Problem 5: 5.20

Problem 5.20 Current I flows along the positive z -direction in the inner conductor of a long coaxial cable and returns through the outer conductor. The inner conductor has radius a , and the inner and outer radii of the outer conductor are b and c , respectively.

- Determine the magnetic field in each of the following regions: $0 \leq r \leq a$, $a \leq r \leq b$, $b \leq r \leq c$, and $r \geq c$.
- Plot the magnitude of \mathbf{H} as a function of r over the range from $r = 0$ to $r = 10$ cm, given that $I = 10$ A, $a = 2$ cm, $b = 4$ cm, and $c = 5$ cm.

Solution:

- Following the solution to Example 5-5, the magnetic field in the region $r < a$,

$$\mathbf{H} = \hat{\phi} \frac{rI}{2\pi a^2},$$

and in the region $a < r < b$,

$$\mathbf{H} = \hat{\phi} \frac{I}{2\pi r}.$$

The total area of the outer conductor is $A = \pi(c^2 - b^2)$ and the fraction of the area of the outer conductor enclosed by a circular contour centered at $r = 0$ in the region $b < r < c$ is

$$\frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)} = \frac{r^2 - b^2}{c^2 - b^2}.$$

The total current enclosed by a contour of radius r is therefore

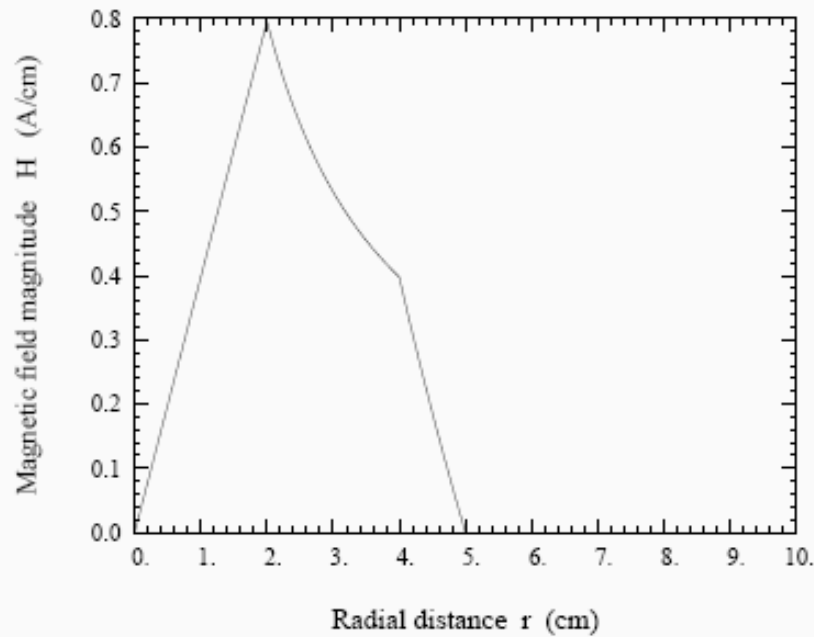
$$I_{\text{enclosed}} = I \left(1 - \frac{r^2 - b^2}{c^2 - b^2} \right) = I \frac{c^2 - r^2}{c^2 - b^2},$$

and the resulting magnetic field is

$$\mathbf{H} = \hat{\phi} \frac{I_{\text{enclosed}}}{2\pi r} = \hat{\phi} \frac{I}{2\pi r} \left(\frac{c^2}{c^2} \frac{r^2}{b^2} \right).$$

For $r > c$, the total enclosed current is zero: the total current flowing on the inner conductor is equal to the total current flowing on the outer conductor, but they are flowing in opposite directions. Therefore, $\mathbf{H} = 0$.

(b) See Fig. P5.20.



Problem 6: 5.26

Problem 5.26 A uniform current density given by

$$\mathbf{J} = \hat{z}J_0 \quad (\text{A/m}^2),$$

gives rise to a vector magnetic potential

$$\mathbf{A} = \hat{z} \frac{\mu_0 J_0}{4} (x^2 + y^2) \quad (\text{Wb/m}).$$

- Apply the vector Poisson's equation to confirm the above statement.
- Use the expression for \mathbf{A} to find \mathbf{H} .
- Use the expression for \mathbf{J} in conjunction with Ampère's law to find \mathbf{H} . Compare your result with that obtained in part (b).

Solution:

(a)

$$\begin{aligned} \nabla^2 \mathbf{A} &= \hat{x} \nabla^2 A_x + \hat{y} \nabla^2 A_y + \hat{z} \nabla^2 A_z = \hat{z} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left[\mu_0 \frac{J_0}{4} (x^2 + y^2) \right] \\ &= \hat{z} \mu_0 \frac{J_0}{4} (2 + 2) = \hat{z} \mu_0 J_0. \end{aligned}$$

Hence, $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ is verified.

(b)

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu_0} \nabla \times \mathbf{A} = \frac{1}{\mu_0} \left[\hat{x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \\ &= \frac{1}{\mu_0} \left(\hat{x} \frac{\partial A_z}{\partial y} - \hat{y} \frac{\partial A_z}{\partial x} \right) \\ &= \frac{1}{\mu_0} \left[\hat{x} \frac{\partial}{\partial y} \left(-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) - \hat{y} \frac{\partial}{\partial x} \left(-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) \right] \\ &= \hat{x} \frac{J_0 y}{2} - \hat{y} \frac{J_0 x}{2} \quad (\text{A/m}). \end{aligned}$$

(c)

$$\begin{aligned} \oint_C \mathbf{H} \cdot d\mathbf{l} &= I = \int_S \mathbf{J} \cdot d\mathbf{s}, \\ \hat{\phi} H_\phi \cdot \hat{\phi} 2\pi r &= J_0 \cdot \pi r^2, \\ \mathbf{H} &= \hat{\phi} H_\phi = \hat{\phi} J_0 \frac{r}{2}. \end{aligned}$$

We need to convert the expression from cylindrical to Cartesian coordinates. From Table 3-2,

$$\begin{aligned} \hat{\phi} &= -\hat{x} \sin \phi + \hat{y} \cos \phi = -\hat{x} \frac{y}{\sqrt{x^2 + y^2}} + \hat{y} \frac{x}{\sqrt{x^2 + y^2}}, \\ r &= \sqrt{x^2 + y^2}. \end{aligned}$$

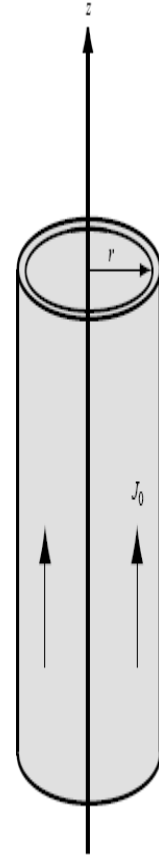


Figure P5.26: Current cylinder of Problem 5.26.

Hence

$$\mathbf{H} = \left(\hat{x} \frac{y}{\sqrt{x^2 + y^2}} \quad \hat{y} \frac{x}{\sqrt{x^2 + y^2}} \right) \cdot \frac{J_0}{2} \sqrt{x^2 + y^2} = \hat{x} \frac{yJ_0}{2} \quad \hat{y} \frac{xJ_0}{2},$$

which is identical with the result of part (b).