HW 7 solution

Problem 4.22 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a, and charge Q_2 is uniformly distributed over a second spherical shell of radius b, with $b > a$. Apply Gauss's law to find $\mathbf{E}$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}} D_R$. From Table 3.1, $d\mathbf{s} = \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi$ for an element of a spherical surface. Using Gauss's law in integral form (Eq. (4.29)),

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{\text{tot}},$$

where Q_{tot} is the total charge enclosed in S. For a spherical surface of radius R,

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\hat{\mathbf{R}} D_R) \cdot (\hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi) = Q_{\text{tot}},$$

$$D_R R^2 (2\pi) \left[-\cos\theta \big|_0^{\pi} \right] = Q_{\text{tot}},$$

$$D_R = \frac{Q_{\text{tot}}}{4\pi R^2}.$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \varepsilon \mathbf{E}$. Thus, we find $\mathbf{E}$ from $\mathbf{D}$.

(a) In the region $R \quad a$,

$$Q_{\text{tot}} = 0, \qquad \mathbf{E} = \hat{\mathbf{R}} E_R = \frac{\hat{\mathbf{R}} Q_{\text{tot}}}{4\pi R^2 \varepsilon} = 0 \quad \text{(V/m)}.$$

(b) In the region $a < R < b$,

$$Q_{\text{tot}} = Q_1, \qquad \mathbf{E} = \hat{\mathbf{R}} E_R = \frac{\hat{\mathbf{R}} Q_1}{4\pi R^2 \varepsilon} \quad \text{(V/m)}.$$

(c) In the region $R > b$,

$$Q_{\text{tot}} = Q_1 + Q_2, \qquad \mathbf{E} = \hat{\mathbf{R}} E_R = \frac{\hat{\mathbf{R}} (Q_1 \quad Q_2)}{4\pi R^2 \varepsilon} \quad \text{(V/m)}.$$

Problem 4.23 The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}} \rho_0 R \quad \text{(C/m}^2\text{)},$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}} \rho_0 R \cdot \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi \Big|_{R=a}$$

$$= 2\pi \rho_0 a^3 \int_0^{\pi} \sin\theta \, d\theta = -2\pi \rho_0 a^3 \cos\theta \big|_0^{\pi} = 4\pi \rho_0 a^3 \quad \text{(C)}.$$

Problem 4.24 In a certain region of space, the charge density is given in cylindrical coordinates by the function:

$$\rho_v = 50re^{-r} \quad (\text{C/m}^3).$$

Apply Gauss's law to find **D**.

Solution:

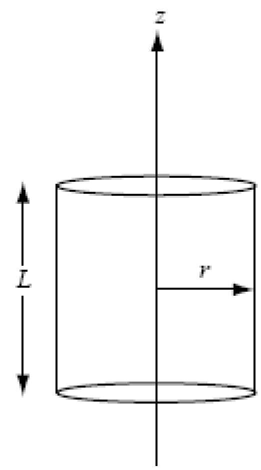

Figure P4.24: Gaussian surface.

Since ρ_v varies as a function of r only, so will **D**. Hence, we construct a cylinder of radius r and length L, coincident with the z-axis. Symmetry suggests that **D** has the functional form $\mathbf{D} = \hat{\mathbf{r}} D$. Hence,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q,$$

$$\int \hat{\mathbf{r}} D \cdot d\mathbf{s} = D(2\pi rL),$$

$$Q = 2\pi L \int_0^r 50re^{-r} \cdot r\,dr$$

$$= 100\pi L[-r^2e^{-r} + 2(1 - e^{-r}(1+r))],$$

$$\mathbf{D} = \hat{\mathbf{r}} D = \hat{\mathbf{r}}\, 50 \left[\frac{2}{r}(1 - e^{-r}(1+r)) - re^{-r}\right].$$

Or using differential form:

$$\nabla \cdot \mathbf{D} = \rho_v, \qquad \mathbf{D} = \hat{\mathbf{r}} D_r,$$

with D_r being a function of r.

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) = 50re^{-r},$$

Then integrate from 0 to r to get the result above.

Problem 4.25 An infinitely long cylindrical shell extending between $r = 1$ m and $r = 3$ m contains a uniform charge density ρ_{v0}. Apply Gauss's law to find $\mathbf{D}$ in all regions.

Solution: For $r < 1$ m, $\mathbf{D} = 0$.
For $1 \le r \le 3$ m,

$$\oint_S \hat{\mathbf{r}} D_r \cdot d\mathbf{s} = Q,$$

$$D_r \cdot 2\pi r L = \rho_{v0} \cdot \pi L (r^2 - 1^2),$$

$$\mathbf{D} = \hat{\mathbf{r}} D_r = \hat{\mathbf{r}} \frac{\rho_{v0} \pi L (r^2 \quad 1)}{2\pi r L} = \hat{\mathbf{r}} \frac{\rho_{v0}(r^2 \quad 1)}{2r}, \qquad 1 \le r \le 3 \text{ m}.$$

For $r \ge 3$ m,

$$D_r \cdot 2\pi r L = \rho_{v0} \pi L (3^2 - 1^2) = 8 \rho_{v0} \pi L,$$

$$\mathbf{D} = \hat{\mathbf{r}} D_r = \hat{\mathbf{r}} \frac{4\rho_{v0}}{r}, \qquad r \ge 3 \text{ m}.$$

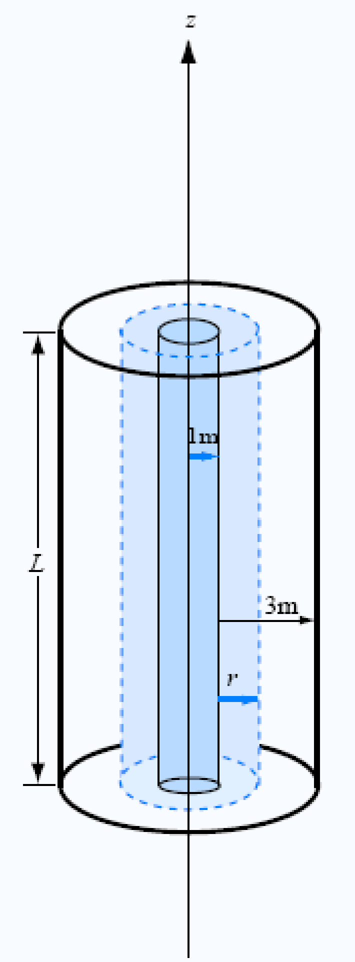

Figure P4.25: Cylindrical shell.