EE331: Homewrok 9 solutions

Problem 1:

Problem 4.20 Given the electric flux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\mathrm{C/m^2}),$$

determine

(a) ρ_v by applying Eq. (4.26),

(b) the total charge Q enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin, and

(c) the total charge Q in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

(b) Integrate the charge density over the volume as in Eq. (4.27):

$$Q = \int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} 0\, dx\, dy\, dz = 0.$$

(c) Apply Gauss' law to calculate the total charge from Eq. (4.29)

$$Q = \oint \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}},$$

$$F_{\text{front}} = \int_{y=0}^{2}\int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=2} \cdot (\hat{\mathbf{x}}\, dz\, dy)$$

$$= \int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\Big|_{x=2}\, dz\, dy = \left(2z\left(2y+\frac{1}{2}y^2\right)\Big|_{z=0}^{2}\right)\Big|_{y=0}^{2} = 24,$$

$$F_{\text{back}} = \int_{y=0}^{2}\int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=0} \cdot (-\hat{\mathbf{x}}\, dz\, dy)$$

$$= \int_{y=0}^{2}\int_{z=0}^{2} 2(x\ \ y)\Big|_{x=0}\, dz\, dy = \left(zy^2\Big|_{z=0}^{2}\right)\Big|_{y=0}^{2} = \ 8,$$

$$F_{\text{right}} = \int_{x=0}^{2}\int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x\ \ y)\ \ \hat{\mathbf{y}}(3x\ \ 2y)\right)\Big|_{y=2} \cdot (\hat{\mathbf{y}}\, dz\, dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=2}\, dz\, dx = \left(z\left(\frac{3}{2}x^2-4x\right)\Big|_{z=0}^{2}\right)\Big|_{x=0}^{2} = -4,$$

$$F_{\text{left}} = \int_{x=0}^{2}\int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x\ \ y)\ \ \hat{\mathbf{y}}(3x\ \ 2y)\right)\Big|_{y=0} \cdot (\ \ \hat{\mathbf{y}}\, dz\, dx)$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=0}\, dz\, dx = -\left(z\left(\frac{3}{2}x^2\right)\Big|_{z=0}^{2}\right)\Big|_{x=0}^{2} = -12,$$

$$F_{\text{top}} = \int_{x=0}^{2}\int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x\ \ y)\ \ \hat{\mathbf{y}}(3x\ \ 2y)\right)\Big|_{z=2} \cdot (\hat{\mathbf{z}}\, dy\, dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=2}\, dy\, dx = 0,$$

$$F_{\text{bottom}} = \int_{x=0}^{2}\int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x\ \ y)\ \ \hat{\mathbf{y}}(3x\ \ 2y)\right)\Big|_{z=0} \cdot (\hat{\mathbf{z}}\, dy\, dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=0}\, dy\, dx = 0.$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 24\ \ 8\ \ 4\ \ 12\ .\ 0\ .\ 0 = 0.$

Problem 2:

Problem 4.30 Show that the electric potential difference V_{12} between two points in air at radial distances r_1 and r_2 from an infinite line of charge with density ρ_l along the z-axis is $V_{12} = (\rho_l/2\pi\varepsilon_0)\ln(r_2/r_1)$.

Solution: From Eq. (4.33), the electric field due to an infinite line of charge is

$$\mathbf{E} = \hat{\mathbf{r}}E_r = \hat{\mathbf{r}}\frac{\rho_l}{2\pi\varepsilon_0 r}.$$

Hence, the potential difference is

$$V_{12} = -\int_{r_2}^{r_1} \mathbf{E}\cdot d\mathbf{l} = -\int_{r_2}^{r_1} \frac{\hat{\mathbf{r}}\rho_l}{2\pi\varepsilon_0 r}\cdot\hat{\mathbf{r}}\,dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right).$$

Problem 3:

Problem 4.34 Given the electric field

$$\mathbf{E} = \hat{\mathbf{R}}\frac{18}{R^2} \quad \text{(V/m)},$$

find the electric potential of point A with respect to point B where A is at $+2$ m and B at -4 m, both on the z-axis.

Solution:

$$V_{AB} = V_A \quad V_B = \quad \int_B^A \mathbf{E}\cdot d\mathbf{l}.$$

Along z-direction, $\hat{\mathbf{R}} = \hat{\mathbf{z}}$ and $\mathbf{E} = \hat{\mathbf{z}}\frac{18}{z^2}$ for $z \geq 0$, and $\hat{\mathbf{R}} = \quad \hat{\mathbf{z}}$ and $\mathbf{E} = \quad \hat{\mathbf{z}}\frac{18}{z^2}$ for $z \leq 0$. Hence,

$$V_{AB} = \quad \int_{-4}^{2} \hat{\mathbf{R}}\frac{18}{z^2}\cdot\hat{\mathbf{z}}\,dz = \quad \left[\int_{-4}^{0} \quad \hat{\mathbf{z}}\frac{18}{z^2}\cdot\hat{\mathbf{z}}\,dz \;.\; \int_0^2 \hat{\mathbf{z}}\frac{18}{z^2}\cdot\hat{\mathbf{z}}\,dz\right] = 4\text{ V}.$$

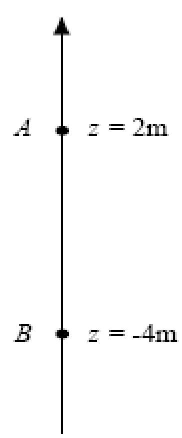

Figure P4.34: Potential between B and A.

Problem 4:

Problem 4.44 An infinitely long conducting cylinder of radius a has a surface charge density ρ_s. The cylinder is surrounded by a dielectric medium with $\varepsilon_r = 4$ and contains no free charges. If the tangential component of the electric field in the region $r \geq a$ is given by $\mathbf{E}_t = -\hat{\boldsymbol{\phi}} \cos^2\phi / r^2$, find ρ_s.

Solution: Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,

$$\mathbf{E}_2 = \hat{\mathbf{r}} E_r - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos^2\phi,$$

with E_r, the normal component of $\mathbf{E}_2$, unknown. The surface charge density is related to E_r. To find E_r, we invoke Gauss's law in medium 2:

$$\nabla \cdot \mathbf{D}_2 = 0,$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(-\frac{1}{r^2}\cos^2\phi\right) = 0,$$

which leads to

$$\frac{\partial}{\partial r}(rE_r) = \frac{\partial}{\partial\phi}\left(\frac{1}{r^2}\cos^2\phi\right) = \quad \frac{2}{r^2}\sin\phi\cos\phi.$$

Integrating both sides with respect to r,

$$\int \frac{\partial}{\partial r}(rE_r)\,dr = \quad 2\sin\phi\cos\phi \int \frac{1}{r^2}\,dr$$

$$rE_r = \frac{2}{r}\sin\phi\cos\phi,$$

or

$$E_r = \frac{2}{r^2}\sin\phi\cos\phi.$$

Hence,

$$\mathbf{E}_2 = \hat{\mathbf{r}}\frac{2}{r^2}\sin\phi\cos\phi \quad \hat{\boldsymbol{\phi}}\frac{1}{r^2}\cos^2\phi.$$

According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 \quad \mathbf{D}_2) = \rho_s,$$

where $\hat{\mathbf{n}}_2$ is the normal to the boundary and points away from medium 1. Hence, $\hat{\mathbf{n}}_2 = \hat{\mathbf{r}}$. Also, $\mathbf{D}_1 = 0$ because the cylinder is a conductor. Consequently,

$$\begin{aligned}
\rho_s &= -\hat{\mathbf{r}} \cdot \mathbf{D}_2|_{r=a} \\
&= \quad \hat{\mathbf{r}} \cdot \varepsilon_2 \mathbf{E}_2|_{r=a} \\
&= \quad \hat{\mathbf{r}} \cdot \varepsilon_r\varepsilon_0 \left[\hat{\mathbf{r}}\frac{2}{r^2}\sin\phi\cos\phi \quad \hat{\boldsymbol{\phi}}\frac{1}{r^2}\cos^2\phi\right]_{r=a} \\
&= -\frac{8\varepsilon_0}{a^2}\sin\phi\cos\phi \quad (\text{C/m}^2).
\end{aligned}$$

Problem 5:

Problem 4.46 If $\mathbf{E} = \hat{\mathbf{R}}150$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

Solution: From Table 4-3, $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. $\mathbf{E}_2$ inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1\text{R}} = \rho_s, \qquad E_{1\text{R}} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

$$Q = E_{\text{R}} S\varepsilon_0 = (150)4\pi(0.05)^2\varepsilon_0 = \frac{3\pi\varepsilon_0}{2} \quad (\text{C}).$$