Problem 1:

Problem 4.20  Given the electric flux density

\[ \mathbf{D} = \varepsilon \hat{x} (x + y) + \varepsilon (3x - 2y) \quad \text{(C/m}^2) \],

determine

(a) \( \rho_v \) by applying Eq. (4.26),
(b) the total charge \( Q \) enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the \( x-, y-, \) and \( z- \) axes and one of its corners at the origin, and
(c) the total charge \( Q \) in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

\[ \rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2x + 2y) + \frac{\partial}{\partial y}(3x - 2y) = 0. \]

(b) Integrate the charge density over the volume as in Eq. (4.27):

\[ Q = \int_V \nabla \cdot \mathbf{D} dV = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} 0 \, dx \, dy \, dz = 0. \]

(c) Apply Gauss' law to calculate the total charge from Eq. (4.29)

\[ Q = \int \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}, \]

\[ F_{\text{front}} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{x} z (x + y) + \hat{y} (3x - 2y)) \cdot (\hat{x} \, dz \, dy) \]
\[ = \int_{x=0}^{2} \int_{z=0}^{2} 2(x + y) \bigg|_{x=2}^{x=0} \, dz \, dy = \left( 2 \left( 2y + \frac{1}{2} y^2 \right) \right) \bigg|_{y=0}^{y=2} = 24, \]

\[ F_{\text{back}} = \int_{y=0}^{2} \int_{z=0}^{2} (\hat{x} z (x + y) + \hat{y} (3x - 2y)) \cdot (-\hat{x} \, dz \, dy), \]
\[ = \int_{y=0}^{2} \int_{z=0}^{2} 2(x + y) \bigg|_{x=0}^{x=2} \, dz \, dy = \left( 2y^2 \right) \bigg|_{y=0}^{y=2} = 8, \]

\[ F_{\text{right}} = \int_{x=0}^{2} \int_{y=0}^{2} (\hat{x} z (x + y) + \hat{y} (3x - 2y)) \cdot (\hat{y} \, dz \, dx), \]
\[ = \int_{x=0}^{2} \int_{y=0}^{2} (3x - 2y) \bigg|_{x=2}^{x=0} \, dz \, dx = \left( 2 \left( \frac{3}{2} x^2 - 4x \right) \right) \bigg|_{x=0}^{x=2} = -4, \]

\[ F_{\text{left}} = \int_{x=0}^{2} \int_{y=0}^{2} (\hat{x} z (x + y) + \hat{y} (3x - 2y)) \cdot (\hat{x} \, dz \, dx), \]
\[ - \int_{y=0}^{2} \int_{z=0}^{2} (3x - 2y) \bigg|_{y=0}^{y=2} \, dz \, dx = -\left( z \left( \frac{3}{2} x^2 \right) \right) \bigg|_{x=0}^{x=2} = -12, \]

\[ F_{\text{top}} = \int_{x=0}^{2} \int_{y=0}^{2} (\hat{x} z (x + y) + \hat{y} (3x - 2y)) \cdot (\hat{z} \, dy \, dx), \]
\[ = \int_{x=0}^{2} \int_{y=0}^{2} 0 \bigg|_{z=2}^{z=0} dy \, dx = 0, \]

\[ F_{\text{bottom}} = \int_{x=0}^{2} \int_{y=0}^{2} (\hat{x} z (x + y) + \hat{y} (3x - 2y)) \cdot (\hat{z} \, dy \, dx), \]
\[ = \int_{x=0}^{2} \int_{y=0}^{2} 0 \bigg|_{z=0}^{z=2} dy \, dx = 0. \]

Thus \[ Q = \int \mathbf{D} \cdot d\mathbf{s} = 24 \quad 8 \quad 12 \quad 0 \quad 0 \quad 0 = 0. \]
Problem 2:

**Problem 4.30** Show that the electric potential difference $V_{12}$ between two points in air at radial distances $r_1$ and $r_2$ from an infinite line of charge with density $\rho_l$ along the $z$-axis is $V_{12} = (\rho_l/2\pi\varepsilon_0) \ln(r_2/r_1)$.

**Solution:** From Eq. (4.33), the electric field due to an infinite line of charge is

$$E = \hat{r}E_r = \hat{r} \frac{\rho_l}{2\pi\varepsilon_0 r}.$$  

Hence, the potential difference is

$$V_{12} = -\int_{r_2}^{r_1} E \cdot d\mathbf{l} = -\int_{r_2}^{r_1} \frac{\rho_l}{2\pi\varepsilon_0 r} \hat{r} dr = \frac{\rho_l}{2\pi\varepsilon_0} \ln \left( \frac{r_2}{r_1} \right).$$

Problem 3:

**Problem 4.34** Given the electric field

$$E = \frac{\hat{R}}{R^2} 18 \text{ V/m},$$

find the electric potential of point $A$ with respect to point $B$ where $A$ is at $+2$ m and $B$ at $-4$ m, both on the $z$-axis.

**Solution:**

$$V_{AB} = V_A - V_B = \int_B^A E \cdot d\mathbf{l}.$$  

Along $z$-direction, $\hat{R} = \hat{z}$ and $E = \frac{18}{z^2}$ for $z \geq 0$, and $\hat{R} = -\hat{z}$ and $E = \frac{18}{z^2}$ for $z \leq 0$. Hence,

$$V_{AB} = \int_{-4}^2 \frac{18}{z^2} \cdot \hat{z} dz = \left[ \int_{-4}^0 \frac{18}{z^2} \cdot \hat{z} dz + \int_{0}^2 \frac{18}{z^2} \cdot \hat{z} dz \right] = 4 \text{ V}.$$

![Figure P4.34: Potential between $B$ and $A$.](image)
Problem 4:

**Problem 4.44** An infinitely long conducting cylinder of radius $a$ has a surface charge density $\rho_s$. The cylinder is surrounded by a dielectric medium with $\varepsilon_r = 4$ and contains no free charges. If the tangential component of the electric field in the region $r \geq a$ is given by $E_t = -\Phi \cos^2 \phi / r^4$, find $\rho_s$.

**Solution:** Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,

$$E_2 = \hat{r}E_t - \hat{r} \frac{1}{r^2} \cos^2 \phi,$$

with $E_t$, the normal component of $E_2$, unknown. The surface charge density is related to $E_t$. To find $E_t$, we invoke Gauss’s law in medium 2:

$$\nabla \cdot D_2 = 0,$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} \left( rE_t \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \frac{1}{r^2 \sin^2 \phi} \right) = 0,$$

which leads to

$$\frac{\partial}{\partial r} \left( rE_t \right) = \frac{1}{r^2 \sin^2 \phi} \left( \frac{1}{r^2} \cos^2 \phi \right) = \frac{2}{r^2} \sin \phi \cos \phi.$$

Integrating both sides with respect to $r$,

$$\int \frac{\partial}{\partial r} \left( rE_t \right) dr = \int \frac{2}{r^2} \sin \phi \cos \phi dr,$$

$$rE_t = \frac{2}{r^2} \sin \phi \cos \phi,$$

or

$$E_t = \frac{2}{r^2} \sin \phi \cos \phi.$$

Hence,

$$E_2 = \hat{r} \frac{2}{r^2} \sin \phi \cos \phi - \hat{r} \frac{1}{r^2} \cos^2 \phi.$$

According to Eq. (4.93),

$$\hat{n}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s,$$

where $\hat{n}_2$ is the normal to the boundary and points away from medium 1. Hence, $\hat{n}_2 = \hat{r}$. Also, $\mathbf{D}_1 = 0$ because the cylinder is a conductor. Consequently,

$$\rho_s = -\hat{r} \cdot \mathbf{D}_2 |_{r=a} = \hat{r} \cdot \varepsilon_0 E_2 |_{r=a} = \hat{r} \cdot \varepsilon_0 \left[ \frac{2}{r^2} \sin \phi \cos \phi \right] |_{r=a} = -\frac{8\varepsilon_0 \varepsilon_0}{a^2} \sin \phi \cos \phi \quad \text{(C/m)}.$$

Problem 5:

**Problem 4.46** If $E = \hat{r} 150 \text{ (V/m)}$ at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge $Q$ on the sphere’s surface?

**Solution:** From Table 4-3, $\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. $E_2$ inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1R} = \rho_s, \quad E_{1R} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

$$Q = E_R S \varepsilon_0 = (150) 4\pi (0.05)^2 \varepsilon_0 = \frac{3\pi \varepsilon_0}{2} \quad \text{(C)}.$$