## Problem 9.23:
In a certain region with $\sigma=0, \mu=\mu_0, \text{and } \varepsilon=6.25\,\varepsilon_0$, the magnetic field on an EM wave is

$$H=0.6\cos\beta x\cos 10^8 t\,\hat{a}_z \text{ A/m}$$

Find $\beta$ and the corresponding **E** using Maxwell's equations.

## Solution:
We are going to use Maxwell's equations to loop back to H. In doing so, we will be able to find $\beta$ and then the **E** field.

$$\varepsilon\frac{\partial E}{\partial t}=\nabla\times H=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & H_z\end{vmatrix}=\frac{-\partial H_z}{\partial x}\hat{a}_y=0.6\beta\sin\beta x\cos\omega t\,\hat{a}_y$$

$$\hat{E}=\frac{1}{\varepsilon}\int(\nabla\times\hat{H})\partial t=\frac{0.6\beta}{\omega\varepsilon}\sin\beta x\sin\omega t\,\hat{a}_y$$

$$\nabla\times\hat{E}=-\mu\frac{\partial\hat{H}}{\partial t}=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & E_y & 0\end{vmatrix}=\frac{0.6\beta^2}{\omega^2\mu\varepsilon}\cos\beta x\cos\omega t\,\hat{a}_z$$

Now we can derive an equation for H for comparison

$$H=-\frac{1}{\mu}\int(\nabla\times E)\partial t=\frac{0.6\beta^2}{\omega^2\mu\varepsilon}\cos\beta x\cos\omega t\,\hat{a}_z$$

From above we have

$$H=0.6\cos\beta x\cos 10^8 t\,\hat{a}_z=\frac{0.6\beta^2}{\omega^2\mu\varepsilon}\cos\beta x\cos\omega t\,\hat{a}_z$$

by cancelling like terms we arrive at the expression:

$1=\frac{\beta^2}{\omega^2\mu\varepsilon}$ and with a little algebra that becomes $\beta=\omega\sqrt{\mu\varepsilon}$

By definition, we know the speed of light *c* to be $c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}=3\times10^8\,m/s$

With this definition, the equation for $\beta$ becomes:

$$\beta=\frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}=\frac{10^8}{3\times10^8}\sqrt{(1)(6.25)}=\mathbf{0.833\,rad/s}$$

Now that we have $\beta$, we can solve for **E**.

$$E_0=\frac{0.6\beta}{\omega\varepsilon}=\frac{0.6\beta}{\omega\varepsilon_r\varepsilon_0}=\frac{(0.6)(0.833)}{(10^8)(6.25)(8.854\times10^{-12})}=93.6$$

$$\boxed{\beta=0.833\text{ m/s and } E=96.3\sin\beta x\sin\omega t\,\hat{a}_y \text{ V/m}}$$

## Problem 9.32: Write the following time-harmonic fields as phasors:

A) $E = 4\cos(\omega t - 3x - 10^o)\hat{a}_y - \sin(\omega t + 3x + 20^o)\hat{a}_z$

$$E = 4\cos(\omega t - 3x - 10^o)\hat{a}_y - \cos(\omega t + 3x - 70^o)\hat{a}_z \quad (recall: \sin\theta = \cos(\theta - 90^o))$$
$$= \mathrm{Re}[4e^{-j(3x+10^o)} e^{j\omega t} e^{-j5r} \hat{a}_y - e^{-j(3x-70^o)} e^{j\omega t} \hat{a}_z] = \mathrm{Re}[H_s e^{j\omega t}]$$
$$\boldsymbol{E_s = 4e^{-j(3x+10^o)}\hat{a}_y - e^{-j(3x-70^o)}\hat{a}_z}$$

B) $H = \frac{\sin\theta}{r}\cos(\omega t - 5r)\hat{a}_\theta$

$$= \mathrm{Re}[\frac{\sin\theta}{r} e^{j\omega t} e^{-j5r}\hat{a}_\theta] = \mathrm{Re}[H_s e^{j\omega t}]$$
$$\boldsymbol{H_s = \frac{\sin\theta}{r} e^{-j5r}\hat{a}_\theta}$$

C) $J = 6e^{-3x}\sin(\omega t - 2x)\hat{a}_y + 10e^{-x}\cos(\omega t - 5x)\hat{a}_z$

$$= \mathrm{Re}[6e^{-3x}e^{-j2x}e^{-j90^o}e^{j\omega t}\hat{a}_y + 10e^{-x}e^{j\omega t}e^{-j5x}\hat{a}_z]$$
$$J_s = 6e^{-(3x+j2x+j90^o)}\hat{a}_y + 10e^{-(1+j5)}\hat{a}_z$$
$$\boldsymbol{= -j6e^{-(3+j2)x}\hat{a}_y + 10e^{-(1+j5)x}\hat{a}_z}$$

## Problem 9.33: Express the following phasors in their instantaneous forms:

A) $A_s = (4-3j)e^{-j\beta x}\hat{a}_y$

$A = \mathrm{Re}[A_s e^{j\omega t}] = (4-3j)e^{-j\beta x}e^{j\omega x}\hat{a}^y$

$(aside: 4-3j = 5\measuredangle 36.87^o = 5e^{-j36.87^o})$

$\boldsymbol{A = 5\cos(\omega t - \beta x - 36.87^o)\hat{a}_y}$

B) $B_s = \frac{20}{\rho}e^{-j2z}\hat{a}_\rho$

$B = \mathrm{Re}[B_s e^{j\omega t}] = \boldsymbol{\frac{20}{\rho}\cos(\omega t - 2z)\hat{a}_\rho}$

C) $C_s = \frac{10}{r^2}(1+j2)e^{-j\phi}\sin\theta\,\hat{a}_\phi = \frac{10}{r^2}(2.24)e^{j63.43^o}e^{-j\phi}\sin\theta\,\hat{a}_\phi$

$(aside: 1+j3 = 2.24\measuredangle 63.43^o = 2.24e^{j63.43^o})$

$C = \mathrm{Re}[C_s e^{j\omega t}] = \mathrm{Re}[\frac{22.4}{r^2}e^{j(\omega t - \phi + 63.43^o)}\sin\theta\,\hat{a}_\phi]$

$= \boldsymbol{\frac{22.4}{r^2}\cos(\omega t - \phi + 63.43^o)\sin\theta\,\hat{a}_\phi}$

## Problem 10.1: An EM wave propagating in a certain medium is described by

$$E = 25\sin(2\pi\times 10^6 t - 6x)\hat{a}_z \text{ V/m}$$

A) Determine the direction of wave propagation

The wave propagates in the positive x direction.

B) Compute the period T, the wavelenth $\lambda$ and the velocity *u*.

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi\times 10^6} = \mathbf{1\,\mu s}$

$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6} = \frac{\pi}{3} = \mathbf{1.047\,m}$

$u = \frac{\omega}{\beta} = \frac{2\pi\times 10^6}{6} = \mathbf{1.047\times 10^6\,m}$

C) Sketch the wave at t = 0, T/8, T/4, T/2

$t=0, E_z = 25\sin(-6x) = \boldsymbol{-25\sin(6x)}$

$t = T/8, E_z = 25\sin(\frac{2\pi}{T}*\frac{T}{8}) = \boldsymbol{25\sin(\frac{\pi}{4} - 6x)}$

$t = T/4, E_z = 25\sin(\frac{2\pi}{T}*\frac{T}{4}) = \boldsymbol{25\cos(6x)}$

$t = T/2, E_z = 25\sin(\frac{2\pi}{T}*\frac{T}{2}) = \boldsymbol{25\sin(6x)}$

**Problem 10.3:** At 50 Mhz, a lossy dielectric material is characterized by $\varepsilon = 3.6\,\varepsilon_0$, $\mu = 2.1\,\mu_0$, and $\sigma = 0.08$ S/m. If $E_s = 6\,e^{-\gamma x}\,\hat{a}_z$ V/m, compute:

A) Find $\gamma$

$$\gamma = \alpha + j\beta = 20.2 + j\,20.6$$

$$\left( aside: \frac{\sigma}{\omega\varepsilon} = \frac{8\times10^{-2}}{(50\times10^{6})(3.6)(8.854\times10^{-12})} = \mathbf{50.2} \right)$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)} = \frac{\omega}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}$$

$$= \frac{2\pi\,50\times10^{6}}{3\times10^{8}}\sqrt{(3.6)(2.1)(50.2-1)} = \mathbf{20.2}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right)} = \frac{\omega}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right)}$$

$$= \frac{2\pi\,50\times10^{6}}{3\times10^{8}}\sqrt{(3.6)(2.1)(50.2+1)} = \mathbf{20.6}$$

$$\gamma = \mathbf{20.2 + j\,20.6}$$

B) Find $\lambda$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{20.6} = \mathbf{0.05\,m}$$

C) Find u

$$u = \frac{\omega}{\beta} = \frac{2\pi\,50\times10^{6}}{20.6} = \mathbf{15.3\times10^{6}\,m/s}$$

D) Find $|\eta|$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{|\eta_0|\sqrt{\frac{\mu_r}{\varepsilon_r}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{120\pi\sqrt{\frac{2.1}{3.6}}}{\sqrt[4]{1+(50.2)^2}} = 40.6$$

$$\theta_\eta = 0.5\tan^{-1}\frac{\sigma}{\omega\varepsilon} = \tan^{-1}50.2 = 44.43^o$$

$$\eta = \mathbf{40.6 \measuredangle 44.43^o}$$

E) Find $H_S$

$$H_S = \frac{\hat{a}_k\times E_s}{\eta} = \hat{a}_x\times\frac{6}{\eta}e^{-\gamma z}\,\hat{a}_z = -\frac{6}{\eta}e^{-\gamma z}\,\hat{a}_y = \mathbf{-0.148\,e^{-j44.43^o}\,e^{-\gamma z}\,\hat{a}_y}$$

Problem 10.10: A uniform wave **in air** has $E=10\cos(2\pi\times10^6 t-\beta z)\hat{a}_z$ V/m

Note: Because we are in air, $\mu_r=\varepsilon_r=1$

A) Calculate $\beta$ and $\lambda$

$$\beta=\frac{\omega}{\mu_0\varepsilon_0}=\frac{\omega}{c}=\frac{2\pi\times10^6}{3\times10^8}=\mathbf{20.9\times10^{-3}\,rad/m}$$

$$\lambda=\frac{2\pi}{\beta}=\frac{2\pi}{20.9\times10^{-3}}=\mathbf{300\,m}$$

B) Sketch the wave at $z=0$ and $\lambda/4$

$$z=0\ \Rightarrow E=\mathbf{10\cos(\omega t)}$$

$$z=\frac{\lambda}{4}\ \Rightarrow E=10\cos\left(\omega t-\left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{2}\right)\right)=10\cos\left(\omega t-\frac{\pi}{2}\right)=\mathbf{10\sin(\omega t)}$$

C) Find **H**

$$H_s=\frac{E_s}{|\eta|}=\frac{10}{\sqrt{\frac{\mu_0}{\varepsilon_0}}}=\frac{10}{120\pi}=\frac{1}{12\pi}=26.5\times10^{-3}$$

$$H=H_s\cos(\omega t-\beta z)\hat{a}_x=0.0265\cos(2\pi\times10^6 t-0.0209z)\hat{a}_x$$

$$=\mathbf{26.5\cos(2\pi\times10^6 t-0.0209z)\hat{a}_x\,mA/m}$$

Extra Problem 1: Two linearly polarized waves propagating in the same diection at the same frequency are given by:

$$E_1(y)=C_1 e^{-j\beta y}\hat{a}_x$$
$$E_2(y)=C_2 e^{-j\beta y}e^{j\theta}\hat{a}_z$$

where $C_1$, $C_2$ and $\theta$ are constants and $\beta=\omega\sqrt{\mu\varepsilon}$ . Find

A) The polarization and direction of propagation of E1 and E2.

E1 is *x-polarized* and propagates in the *y* direction
E2 is *z-polarized* and propagates in the *y* direction

B) The polarization of the sum of these two waves in the following cases:

i) $\theta=0$

$$E=E_1+E_2=C_1e^{-j\beta y}\hat{a}_x+E_2(y)+C_2e^{-j\beta y}e^{j0}\hat{a}_z$$
$$=C_1e^{-j\beta y}\hat{a}_x+E_2(y)+C_2e^{-j\beta y}\hat{a}_z$$
$$=(C_1\hat{a}_x+C_2\hat{a}_z)e^{-j\beta y}\rightarrow \boldsymbol{E(y,t)}=(\boldsymbol{C_1\hat{a}_x+C_2\hat{a}_z})\mathbf{\cos}(\omega \boldsymbol{t}-\beta \boldsymbol{y})$$

E is linearly polarized along the vector $C_1\hat{a}_x+C_2\hat{a}_z$ with $\measuredangle\alpha$ at the x=0 plane. $(\tan\alpha=C1/C2)$

Extra Problem 1 (cont): Two linearly polarized waves propagating in the same diection at the same frequency are given by:

$$E_1(y)=C_1 e^{-j\beta y}\hat{a}_x$$
$$E_2(y)=C_2 e^{-j\beta y} e^{j\theta}\hat{a}_z$$

where $C_1$, $C_2$ and $\theta$ are constants and $\beta=\omega\sqrt{\mu\varepsilon}$ . Find

ii) $\theta=\frac{\pi}{2}$

$$E=E_1+E_2=C_1 e^{-j\beta y}\hat{a}_x+E_2(y)+C_2 e^{-j\beta y} e^{j\frac{\pi}{2}}\hat{a}_z$$

take the real part to find ...

$$E_1(y,t)=C_1\cos(\omega t-\beta y)\hat{a}_x$$
$$E_2(y,t)=C_2\cos(\omega t-\beta y+\pi/2)\hat{a}_z=-C_2\sin(\omega t-\beta y)\hat{a}_z$$
$$E(y,t)=C_1\cos(\omega t-\beta y)\hat{a}_x-C_2\sin(\omega t-\beta y)\hat{a}_z$$

Now we have the following relations

$$\frac{E_x^2}{C_1^2}=\frac{E_1^2(y,t)}{C_1^2}=\cos^2(\omega t-\beta y)\hat{a}_x$$
$$\frac{E_z^2}{C_2^2}=\frac{E_2^2(y,t)}{C_2^2}=\sin^2(\omega t-\beta y)\hat{a}_z$$

Thus we have an ellipse (elliptical polarization) defined by:

$$\boldsymbol{\frac{E_1^2(y,t)}{C_1^2}+\frac{E_2^2(y,t)}{C_2^2}=1}=\cos^2(\omega t-\beta y)+\sin^2(\omega t-\beta y)$$

iii) $\theta=\frac{\pi}{2}$ and $C_1=C_2=C$

Using the solution in part ii, we have a circle of radius C (circular polarization)

$$\frac{E_1^2(y,t)}{C_1^2}+\frac{E_2^2(y,t)}{C_2^2}=1\rightarrow \boldsymbol{E_1^2(y,t)+E_2^2(y,t)=C^2}$$

iv) $\theta=\pi$

This is similar to case i, except the C2 component has a 180 degree phase shift.

$$E=E_1+E_2=C_1 e^{-j\beta y}\hat{a}_x+E_2(y)+C_2 e^{-j\beta y} e^{j180^o}\hat{a}_z$$
$$=C_1 e^{-j\beta y}\hat{a}_x+E_2(y)-C_2 e^{-j\beta y}\hat{a}_z$$
$$=\boldsymbol{(C_1\hat{a}_x-C_2\hat{a}_z)e^{-j\beta y}}$$

E is linearly polarized along the vector $C_1\hat{a}_x+C_2\hat{a}_z$ with $\sphericalangle\alpha$ relative to the x=0 plane. $(\tan\alpha=C1/C2)$