Practice Problem 12.1: Consider the waveguide of Example 12.1. Calculate the phase constant, phase velocity and wave impedance for $TE_{10}$ and $TM_{11}$ modes at the operating frequency of 15 GHz.

A) $TE_{10}$

$$\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{1 - \left(\frac{3}{15}\right)^2} = \sqrt{0.96} = 0.9798, \beta = \frac{\omega}{u_0} = 2\pi f/c$$

$$\beta = \beta_0 \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 2\pi \frac{15 \times 10^9}{3 \times 10^8} (0.9798) = 615.6 \text{rad/s} = \beta$$

$$u = \frac{\omega}{\beta} = 2\pi \frac{15 \times 10^9}{615.6} = 153 \times 10^6 \text{m/s} = u$$

$$\eta' = \sqrt{\frac{u}{\varepsilon}} = 60\pi, \eta_{TE} = \frac{60\pi}{\sqrt{0.9798}} = 192.4 \Omega = \eta_{TE}$$

B) $TM_{11}$, $f_c = 3 \text{GHz} \times \sqrt{7.25} = 8.08 \text{GHz}$

$$\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{1 - \left(\frac{8.08}{15}\right)^2} = 0.8425, \beta = \frac{\omega}{u_0} = 4\pi f/c$$

$$\beta = \beta_0 \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 2\pi \frac{15 \times 10^9}{3 \times 10^8} (0.8425) = 529.4 \text{rad/s} = \beta$$

$$u = \frac{\omega}{\beta} = 2\pi \frac{15 \times 10^9}{529.3} = 178 \times 10^6 \text{m/s} = u$$

$$\eta' = \sqrt{\frac{u}{\varepsilon}} = 60\pi, \eta_{TE} = \frac{60\pi}{\sqrt{0.8425}} = 158.8 \Omega = \eta_{TE}$$

Practice Problem 12.2: An air-filled 5- by 2-cm waveguide has $E_z = 20 \sin 40\pi x \sin 50\pi y e^{-j\beta z} \sqrt{\varepsilon/m}$ at 15GHz.

A) What mode is being propagated?

Since $E_z \neq 0$, this is a TM mode

$$E_z = E_0 \sin \left(\frac{m \pi x}{a}\right) \sin \left(\frac{n \pi y}{b}\right) e^{-j\beta z}$$

$$E_0 = 20, \frac{m \pi}{a} = \frac{40 \pi}{0.05 \text{meters}} \rightarrow m = 2, \frac{n \pi}{b} = \frac{50 \pi}{0.02 \text{meters}} \rightarrow n = 1$$

$TM$ mode $= TM_{21}$
Practice Problem 12.2 Continued:

B) Find $\beta$

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3 \times 10^8}{2} \sqrt{40^2 + 50^2} = 9.6 \text{ GHz} = f_c$$

$$\beta = \frac{\omega}{u} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \frac{2 \pi f}{c} \sqrt{f^2 - f_c^2} = \frac{2 \pi \times 10^9}{3 \times 10^8} \sqrt{225 - 92.25}$$

$$\beta = 241.3 \text{ rad/m}$$

C) Determine $E_y/E_z$

$$E_{x0} = -\frac{j \beta}{h^2} (40 \pi) 20 \cos(40 \pi x) \sin(50 \pi y) e^{-j \beta z}$$

$$E_{y0} = -\frac{j \beta}{h^2} (50 \pi) 20 \sin(40 \pi x) \cos(50 \pi y) e^{-j \beta z}$$

$$E_{y1} = -\frac{j \beta}{h^2} (50 \pi) 20 \cos(40 \pi x) \sin(50 \pi y) e^{-j \beta z}$$

$$E_{x1} = -\frac{j \beta}{h^2} (40 \pi) 20 \sin(40 \pi x) \cos(50 \pi y) e^{-j \beta z}$$

$$\frac{E_y}{E_x} = 1.25 \tan(40 \pi x) \cot(50 \pi y)$$

Problem 12.3: A 1-cm × 2-cm waveguide is filled with deionized water with $\varepsilon_r = 81$. If the operating frequency is 4.5GHz, determine:

A) all possible propagating modes

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3 \times 10^8}{2} \sqrt{\left(\frac{m}{2}\right)^2 + \left(\frac{n}{1}\right)^2} = \frac{15}{18} (\sqrt{m^2 + 4n^2}) \times 10^9$$

$$f_c = \frac{5}{6} (\sqrt{m^2 + 4n^2}) \times 10^9 \text{ Hz}$$

<table>
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<th>Frequency (GHz)</th>
<th>$N = 0$</th>
<th>$N = 1$</th>
<th>$N = 2$</th>
</tr>
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<tr>
<td>$M = 0$</td>
<td>DNE</td>
<td>1.667</td>
<td>3.333</td>
</tr>
<tr>
<td>$M = 1$</td>
<td>0.833</td>
<td>1.863</td>
<td>3.436</td>
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<tr>
<td>$M = 2$</td>
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<td>2.357</td>
<td>3.727</td>
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<tr>
<td>$M = 3$</td>
<td>2.500</td>
<td>3.004</td>
<td>4.167</td>
</tr>
<tr>
<td>$M = 4$</td>
<td>3.333</td>
<td>3.727</td>
<td>Too high</td>
</tr>
<tr>
<td>$M = 5$</td>
<td>4.167</td>
<td>4.488</td>
<td>Too high</td>
</tr>
</tbody>
</table>
Problem 12.3 Continued:
B) The intrinsic impedance of the highest mode
The highest possible mode is $TE_{51}$ or $TM_{51}$.

$$\eta' = 120 \frac{\pi}{9} = 41.89, \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{1 - \left(\frac{4.488}{4.5}\right)^2} = 0.073$$

$$\eta_{TE_{51}} = \eta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 41.89 \times 0.073 = 573.8 \Omega = \eta_{TE_{51}}$$

$$\eta_{TM_{51}} = \eta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 41.89 \times 0.073 = 3.058 \Omega = \eta_{TM_{51}}$$

C) The group velocity of the lowest mode
The lowest possible mode is $TE_{10}$.

$$u' = \frac{c}{9}, u_g = u' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 3 \times 10^8 \sqrt{1 - \left(\frac{0.833}{4.5}\right)^2} = 32.76 \times 10^6 m/s = u_g$$

Problem 12.5: A tunnel is modeled as an air-filled metallic rectangular waveguide with dimensions $a = 8m$ and $b = 16m$. Determine whether the tunnel will pass:

For the dominant mode $TE_{01}$, $f_c = \frac{c}{2b} = \frac{3 \times 10^8}{2 \times 16} = 9.375 MHz$

A) a 1.5-MHz AM broadcast signal
The AM signal will not pass because it is below the cutoff frequency

B) a 120-MHz FM broadcast signal
The FM signal will pass because it is above the cutoff frequency
Problem 12.9: An air-filled rectangular waveguide had cross-sectional dimensions $a = 6\text{cm}$ and $b = 3\text{cm}$. Given that

$$E_z = 5 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right) \cos(10^{12} t - \beta z) \text{V/m}$$

Calculate the intrinsic impedance of this mode and the average power flow in the guide.

Since $E_z \neq 0$ this must be $TM_{23}$ mode (m=2, n=3).

Since $a = 2b$,

$$f_c = \frac{c}{4b} \sqrt{m^2 + 4n^2} = \frac{3 \times 10^8}{4 \times 3 \times 10^{-2} \sqrt{4 + 36}} = 15.81 \text{GHz}$$

$$f = \frac{\omega}{2\pi} = \frac{10^{12}}{2\pi} = 159.2 \text{GHz}$$

$$\eta_{TM} = 120\pi \sqrt{1 - \left(\frac{15.81}{159.2}\right)^2} = -375.1 \Omega = \eta_{TM}$$

$$P_{\text{ave}} = \frac{\beta^2 E_0^2}{2h^4 \eta_{TM}} \left[ \left(\frac{2\pi}{a}\right)^2 \cos^2 \left(\frac{2\pi x}{a}\right) \sin^2 \left(\frac{3\pi y}{b}\right) + \left(\frac{3\pi}{b}\right)^2 \cos^2 \left(\frac{2\pi x}{a}\right) \sin^2 \left(\frac{3\pi y}{b}\right) \right] a_z$$

$$P_{\text{ave}} = \int \int_{x=0}^{a} \int_{y=0}^{b} P_{\text{ave}} \ dx \ dy = \frac{\beta^2 E_0^2}{2h^4 \eta_{TM}} \frac{1}{4} \left[ \frac{4\pi^2}{a^2} + \frac{9\pi^2}{b^2} \right] = \frac{\beta^2 E_0^2}{8 h^2 \eta_{TM}}$$

$$\beta = \frac{\omega}{c} \sqrt{1 - \left(\frac{f}{f_c}\right)^2} = \frac{10^{12}}{3 \times 10^8} \sqrt{1 - \left(\frac{15.81}{159.2}\right)^2} = 3.317 \times 10^3$$

$$h^2 = \frac{4\pi^2}{a^2} + \frac{9\pi^2}{b^2} = \frac{10\pi^2}{b^2} = 1.098 \times 10^5$$

$$P_{\text{ave}} = \left(\frac{3.317 \times 10^3}{8}(25)\right) = 0.8347 \text{W} = P_{\text{ave}}$$
Problem 12.10: In an air-filled rectangular waveguide, a TE mode operating at 6GHz has

\[ E_y = 5 \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \cos (\omega t - 12z) \text{V/m} \]

Determine:

A) the mode of operation
Since \( m = 2 \) and \( n = 1 \), we have \( TE_{21} \).

B) the cutoff frequency
\[
\beta = \beta' \sqrt{1 - \left( \frac{f_c}{f} \right)^2} = \frac{\omega}{c} \sqrt{1 - \left( \frac{\omega}{\omega_c} \right)^2}
\]
\[
\beta_c = \sqrt{\omega^2 - \omega_c^2} = \sqrt{\omega^2 - \beta^2 c^2}
\]
\[
f_c = \frac{\omega}{2\pi} = \sqrt{\frac{f^2 - \left( \frac{\beta c}{2\pi} \right)^2}{(6 \times 10^9)^2 - \left( \frac{12 \times 3 \times 10^8}{2\pi} \right)^2}} = 5.973 \text{GHz} = f_c
\]

C) the intrinsic impedance
\[
\eta_{TE} = \frac{\eta}{\sqrt{1 - \left( \frac{f}{f_c} \right)^2}} = \frac{120\pi}{\sqrt{1 - \left( \frac{5.973}{6} \right)^2}} = 3978 \Omega = \eta_{TE}
\]

D) \( H_x \)
For TE mode
\[
E_y = \frac{\omega \mu}{h^2} \left( \frac{m \pi}{a} \right) H_0 \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin (\omega t - \beta z)
\]
\[
H_x = -\frac{\beta}{h^2} \left( \frac{m \pi}{a} \right) H_0 \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin (\omega t - \beta z)
\]

\( \beta = 12, m = 2, n = 1 \)
\[
E_{oy} = \frac{\omega \mu}{h^2} \left( \frac{m \pi}{a} \right) H_0
\]
\[
E_{oy} = \frac{5}{4\pi^2 \times 10^2} = 3.948 \times 10^3
\]
\[
H_{0x} = \frac{E_{oy}}{\eta_{TE}} = \frac{5}{4\pi^2 \times 10^2} = 1.267 \text{mA/m}
\]
\[
H_x = 1.267 \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin (\omega t - \beta z) \text{mA/m}
\]
Problem 12.11: In an air-filled rectangular waveguide with $a = 2.286\text{cm}$ and $b = 1.016\text{cm}$, the $y$-component of the TE mode is given by

$$E_y = \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{3\pi y}{b}\right)\cos(10\pi \times 10^{10} t - \beta z) \text{ V/m}$$

Find:

A) the operating mode
Since $m = 2$ and $n = 3$, the mode is $TE_{23}$.

B) the propagation constant $\gamma$
$$\beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \frac{2\pi f}{c} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

But, $f_c = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3 \times 10^8}{2 \times 10^{-2}} \sqrt{\left(\frac{2}{2.286}\right)^2 + \left(\frac{3}{1.016}\right)^2} = 46.19 \text{ GHz}, f = 50 \text{ GHz}$

$$\beta = \frac{2\pi \times 50 \times 10^9}{3 \times 10^8} \sqrt{1 - \left(\frac{46.19}{50}\right)^2} = 400.68 \text{ rad/m}$$

$$\gamma = j\beta = j400.7/m$$

C) the intrinsic impedance $\eta$
$$\eta = \frac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{120\pi}{\sqrt{1 - \left(\frac{46.19}{50}\right)^2}} = 985.3 \Omega = \eta$$