Given a single valued function $f(x)$ as shown in the figure above, the goal is to determine the integral

$$ I = \int_{x_1=0}^{x_2=x_{\text{max}}} f(x) \, dx $$

The above integral is the area under the curve represented by a solid line in the above figure.

In order to use the Monte method, we need two parameters:

(I) Range of integration. In the above case it runs from $x_1 = 0$ to $x_2 = x_{\text{max}}$. Therefore the full range of integration:

$$ x_2 - x_1 = x_{\text{max}} - 0 = x_{\text{max}} $$

(II) Maximum value of the function $f(x)$ in the range of integration: $f_{\text{max}}$. Values larger than the exact $f_{\text{max}}$ are acceptable.
Numerical Integration Using Monte Carlo Method

\[ I = \int_{x_1=0}^{x_2=x_{\text{max}}} f(x)dx \]

The parameters \( f_{\text{max}} \) and \( x_{\text{max}} \) define the sides of a rectangle as shown above.

The area of the rectangle is given by:

\[ \text{Area} = A = f_{\text{max}} \times x_{\text{max}} \]

The integral \( I \) of the function \( f(x) \) is part of the rectangle defined by \( f_{\text{max}} \) and \( x_{\text{max}} \).

Using Monte Carlo to perform the integration amounts to generating a random sequence of points \((x_r, f_r)\) and checking to see if the points are under the curve defined by \( f(x) \) or not.

1. generate a pair of random numbers \( r_1 \) and \( r_2 \). Note that: \( 0 \leq r_1 \leq 1 \) and \( 0 \leq r_2 \leq 1 \)
2. Calculate \( x_r = r_1 \times x_{\text{max}} \) and \( f_r = r_2 \times f_{\text{max}} \)
3. Check if the point is under the curve. Check if \( f_r \leq f(x_r) \)
4. If the condition in step (3) is true, then accept the point and update the counter for points under curve \( (N_{\text{accept}}) \).

Note that out of the three points in the above figure only point (3) falls below the curve. For points (1) and (2) \( f_r \) does not satisfy the condition.
5. Repeat steps (1) through (4) large number of times \( (N_{\text{trials}}) \).

Typical values of \( N_{\text{trials}} \) range from 10,000 to 1,000,000.

6. Compute the integral \( I = \) (Area under the curve):

\[ I = \frac{N_{\text{accept}}}{N_{\text{trials}}} \times (f_{\text{max}} \times x_{\text{max}}) \]

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Example:
Evaluate the following integral using Monte Carlo method.

\[ I = \int_{\pi}^{0} \cos^2 \theta d\theta \]

This can be evaluated analytically and results in

\[ I = I_{\text{actual}} = \frac{\pi}{2} = 1.57 \]

Solution: We first determine the range of integration and the maximum value \( f_{\text{max}} \)

1. Range of integration: \( \pi - 0 = \pi \) \( x_{\text{max}} = \pi \)
2. Maximum value of the function \( f(\theta) = \cos^2 \theta \) \( f_{\text{max}} = 1 \)
   (You may also use \( f_{\text{max}} \) greater than 1)

The number of trials was varied from 1,000 to 1,000,000.
The error in the integration was also calculated.

<table>
<thead>
<tr>
<th>N_trials</th>
<th>( I_{\text{Monte}} )</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000</td>
<td>1.529955</td>
<td>2.60</td>
</tr>
<tr>
<td>10,000</td>
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<td>100,000</td>
<td>1.566586</td>
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<td>1,000,000</td>
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<td>0.07</td>
</tr>
</tbody>
</table>
Numerical Integration Using Monte Carlo Method

Random Number generation program:
(1) SERRN: sets the sequence of random numbers using a real number Q
(2) Rannum: generates random numbers r uniformly distributed in 0<r<1.

C----- set random number sequence using Q as seed.
C-----
SUBROUTINE SETRN(Q)
IMPLICIT REAL*8(A-H,O-Z)
common /RANDY/ QA1,QA2,QB1,QB2,QBASE
C* INITIALIZE WITH A CALL TO SETRN(0.D0-1.D0)
QA1=2057713.0D0
QA2=16676923.0D0
QBASE=2.**24
QC=DINT(QBASE*(QBASE*Q))
QB1=DINT(QC/QBASE)
QB2=QC-QB1*QBASE
QB1=DMOD(QB1,QBASE)
QB2=DINT(QB2/2.D0)*2.D0 + 1.D0
RETURN
END

Numerical Integration Using Monte Carlo Method

C********************************************************
C-- Random number generator
C--
FUNCTION RANNUM(I)
IMPLICIT REAL*8(A-H,O-Z)
common /RANDY/ QA1,QA2,QB1,QB2,QBASE
C*** FROM CLAMPS AT NRCC - FROM KALOS
10 QD2=QA2*QB2
QE2=DINT(QD2/QBASE)
QC2=QD2-QBASE*QE2
QB1=DMOD(QE2+DMOD(QA1*QB2,QBASE)+DMOD(QA2*QB1,QBASE),QBASE)
QB2=QC2
RANNUM=QB1/QBASE
IF(RANNUM.EQ.0.0D0)GO TO 10
RETURN
END

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Problems:
Determine the following integrals using Monte Carlo method
(write a Matlab code or C):

1) \[ \int_{0}^{\pi} \sin^2(3\theta) \cos^2 \theta \, d\theta \]

2) \[ \int_{0}^{10} \frac{x^3}{x^4 + 16} \, dx \]

(plot and find an approximate maximum and add 0.1 to it). \( I_{\text{actual}} = 1.60984 \). Estimate the error (%).

3) \[ \int_{0}^{\pi} \sin^4 (3x) \, dx \]