EE351 - Spring 2006 - HW4 Solutions

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Problem 1

In free space, a 3 GHz plane wave with the electric field vector given by:
$\bar{E}_i(y,z) = E_{m0}\mathrm{e}^{-\jmath\beta_1(y\cos\theta_i - z\sin\theta_i)}\hat{a}_x$ V/m
is obliquely incident on the $y = 0$ interface between air and polystyrene ($\sigma_2 = 0, \epsilon_{2r} = 2.56, \mu_{2r} = 1$) as shown below. If the time average power is 1.4 W/m^2 and the angle of incidence is 58°, determine the following:

(i) Amplitude of the electric field E_{m0}, propagation vector magnitude β_1

$|P_{ave}| = \frac{\bar{E}_{m0}^2}{2\eta_1} = 1.4$ W/m^2; $\eta_1 = 377$

$\mathbf{\bar{E}_{m0}} = \sqrt{\mathbf{2\eta_1 * 1.4}} = \mathbf{32.5}$ **V**

$\beta_\mathbf{1} = \omega\sqrt{\mu_\mathbf{1}\epsilon_\mathbf{1}} = \frac{\omega}{\mathbf{c}} = \frac{\mathbf{2\pi\times3\times10^9}}{\mathbf{3\times10^8}} = \mathbf{20\pi}$ **rad/m**

$\bar{\beta}_1 = \beta_1\cos\theta_i\hat{a}_x - \beta_1\sin\theta_i\hat{a}_z; \hat{a}_{ki} = \frac{\bar{\beta}_1}{\beta_1} = \cos\theta_i\hat{a}_x - \sin\theta_i\hat{a}_z$

(ii) Incident magnetic field $H_i(y,z)$

$\bar{H}_i = \frac{1}{\eta_1}\hat{a}_{ki}\times\bar{E}_i(y,z) = \frac{1}{\eta_1}(\cos\theta_i\hat{a}_x - \sin\theta_i\hat{a}_z)\times\hat{a}_x E_{m0}\mathrm{e}^{-\jmath\beta_1(y\cos\theta_i - z\sin\theta_i)}$

$\bar{H}_i = \frac{-E_{m0}}{\eta_1}[\cos\theta_i\hat{a}_z + \sin\theta_i\hat{a}_y]\,\mathrm{e}^{-\jmath\beta_1(y\cos\theta_i - z\sin\theta_i)}$

$\mathbf{\bar{H}_i} = \mathbf{-86.2}\,[\mathbf{0.848\hat{a}_y} + \mathbf{0.53\hat{a}_z}]\,\mathrm{e}^{-\jmath\mathbf{20\pi(0.53y - 0.848z)}}$ **mA/m**

(iii) Reflected electric field

To find $\Gamma_\perp$, we need $\theta_t \rightarrow$ Use Snell's Law ($n_1\sin\theta_i = n_2\sin\theta_t$)

$sin58° = \sqrt{2.56}\sin\theta_t \Rightarrow \theta_t = 32°$

$\Gamma_\perp \simeq \frac{cos58° - \sqrt{2.56}\cos 32°}{cos58° + \sqrt{2.56}\cos 32°} = -0.438$

$\bar{E}_r = \Gamma_\perp E_{m0}\mathrm{e}^{\jmath\beta_1(4\cos\theta_r + 2\sin\theta_r)}\hat{a}_x$

$\mathbf{\bar{E}_r} = \mathbf{-14.2e^{\jmath20\pi(0.53y+0.848z)}\hat{a}_x}$ V/m

(iv) Reflected magnetic field

$\hat{a}_E \times \hat{a}_H = \hat{a}_{kr} \Rightarrow \hat{a}_H = \hat{a}_{kr} \times \hat{a}_E$

$\hat{a}_{kr} \times \hat{a}_E = [-0.53\hat{a}_y - 0.848\hat{a}_z] \times [-\hat{a}_x] = [0.53\hat{a}_z - 0.848\hat{a}_y]$

$\mathbf{\bar{H}_r = -37.8\,[0.848\hat{a}_y - 0.53\hat{a}_z]\,e^{\jmath 20\pi(0.53y - 0.848z)}\ mA/m}$

(v) Time average reflected power

$\mathbf{P^r_{ave} = \frac{1}{2}\Re\,[E_r \times H^*_r] = \frac{E_r^2}{2\eta_1} = \frac{14.2^2}{2(377)} = 0.269\ W/m^2}$

(vi) Time average transmitted power

$P^t_{ave} = \frac{E_t^2}{2\eta_2} = \frac{|\tau_\perp E_i|^2}{2\eta_2}$

$\tau_\perp = \frac{2\cos 58^\circ}{\cos 58^\circ + \sqrt{2.56}\cos 32^\circ} = 0.562 \Leftrightarrow (1 + \Gamma_\perp = \tau_\perp)$

$\mathbf{P^t_{ave} = \frac{(0.562 * 32.5)^2}{2 * 236} = 0.707\ W/m^2}$

Problem 2

A 10 W/m^2, 200 Mhz uniform plane wave propagating in a lossless medium has an electric field vector given by:
$\bar{E}_i\left(y,z\right) = \frac{E_m}{\sqrt{2}}\mathrm{e}^{-\jmath\sqrt{\pi}(x+y)}\hat{a}_x - \frac{E_m}{\sqrt{2}}\mathrm{e}^{-\jmath\sqrt{\pi}(x+y)}\hat{a}_y$ V/m
is obliquely incident upon a perfectly conducting surfacve located in the xz plane as shown below. Determine the following:

(i) The propagation vector β_1
$\bar{\beta}_1 \cdot \bar{r} = x\sqrt{\pi} + y\sqrt{\pi} = \beta_{1x}x + \beta_{1y}y + \beta_{1z}z$
$\beta_{1x} = \sqrt{\pi}; \beta_{1y} = \sqrt{\pi}; \beta_{1z} = 0$
$\bar{\beta}_1 = \sqrt{\pi}\hat{a}_x + \sqrt{\pi}\hat{a}_y; \boldsymbol{\beta_1 = \sqrt{2\pi}}$ **rad/m**

(ii) The direction of propagation
$\hat{a}_{ik} = \frac{\bar{\beta}_1}{\beta_1} = \mathbf{\frac{1}{\sqrt{2}}\hat{a}_x + \frac{1}{\sqrt{2}}\hat{a}_y}$

(iii) The angle of incidence θ_i
$\hat{a}_n = \hat{a}_y; cos\theta_i = \hat{a}_{ki} \cdot \hat{a}_n = \frac{1}{\sqrt{2}}\hat{a}_x + \frac{1}{\sqrt{2}}\hat{a}_y \cdot \hat{a}_y = \frac{1}{\sqrt{2}}$
$\boldsymbol{\theta_i = 45^\circ}$

(iv) The dielectric constant of the lossless medium
$\beta_1 = \omega\sqrt{\mu_1\epsilon_1} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_r}$
$\sqrt{\epsilon_r} = \frac{\beta_1}{\omega\sqrt{\mu_0\epsilon_0}} = \frac{\sqrt{2\pi}\times 3\times 10^8}{2\times 10^8\times 2\pi} \Rightarrow$
$\boldsymbol{\epsilon_r = \frac{9\times 2\pi}{4\times(2\pi)^2} = \frac{9}{8\pi} = 0.36}$ (Note: Not realistic number)

(v) Amplitude of the electric field E_m
$P_{ave} = \frac{\left|\bar{E}_i\right|^2}{2\eta_1} = \frac{\left|\frac{\bar{E}_m}{\sqrt{2}}\right|^2}{2\eta_1} + \frac{\left|\frac{\bar{E}_m}{\sqrt{2}}\right|^2}{2\eta_1} = \frac{\left|\bar{E}_m\right|^2}{2\eta_1} = 10$ W/m^2
$E_m = \sqrt{2\eta_1 * 10}; \eta_1 = \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{377}{0.6} = 628.3$
$\mathbf{E_m = 112.1}$ **V**

(vi) The reflected electric field
$\Gamma_{\parallel} = -1$, medium 2 is metal.
$E_y = E_{i\parallel}$; $E_{xi} + E_{xr} = 0$
$E_{xi} = \frac{E_m}{\sqrt{2}}\mathrm{e}^{-\jmath\sqrt{\pi}(x+y)}\hat{a}_x$; $E_{xr} = -\frac{E_m}{\sqrt{2}}\mathrm{e}^{-\jmath\sqrt{\pi}(x-y)}\hat{a}_x$
$\mathbf{E_r} = -\frac{\mathbf{E_m}}{\sqrt{\mathbf{2}}}\mathrm{e}^{-\jmath\sqrt{\pi}(\mathbf{x-y})}\mathbf{\hat{a}_x} - \frac{\mathbf{E_m}}{\sqrt{\mathbf{2}}}\mathrm{e}^{-\jmath\sqrt{\pi}(\mathbf{x-y})}\mathbf{\hat{a}_y}$ V/m

(vii) Polarization of the incident and reflected waves
Both the incident and reflected waves are **linearly polarized**. There is 0 phase difference between the $\hat{a}_x$ and $\hat{a}_y$ components.

Problem 3

Determine the refractive index and minimum thickness of a film to be deposited on the glass surface ($n_3 = 1.52$) such that no normally incident visible light from free space ($\lambda = 550$nm) is reflected. What will be the reflection coefficient if magnesium flouride ($n = 1.38$) was used instead of the film.

Refractive index n_2 & minimum thickness d

$n = \frac{c}{v_p} = \frac{\sqrt{\mu_0 \epsilon}}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\epsilon_r}$

Air $\rightarrow \epsilon = \epsilon_0; \eta_1 = \eta_0; \eta_3 = \frac{\eta_0}{\sqrt{\epsilon_r}}$

$\eta_2 = \sqrt{\eta_1 \eta_3}$

$n_1 = 1; n_3 = \sqrt{\epsilon_{r3}} \Rightarrow \epsilon_{r3} = n_3^2 = 2.31$

$\eta_3 = \frac{\mu}{\epsilon_3} = \frac{\eta_0}{\sqrt{\epsilon_{r3}}} = \frac{\eta_0}{n_3} = \mathbf{248\Omega} = \eta_{\mathbf{3}}$

$\eta_{\mathbf{2}} = \sqrt{\mathbf{337 \times 248}} = \mathbf{305.8\Omega} = \frac{\eta_0}{n_2}$

$n_2 = \frac{\eta_0}{\eta_2} = \frac{377}{305.8} = \mathbf{1.23 = n_2}$

$d = \frac{\lambda_2}{4}$ visible light has $\lambda = 550$ nm in free space

$\lambda_2 = \frac{\lambda_1}{n_2} = \frac{550 \times 10^{-9}}{1.23} = 0.448\mu$m

$\mathbf{d = \frac{0.448}{4} = 0.112\mu m}$

Using magnesium flouride ($n = 1.38 \neq 1.23$), assume $d = \frac{\lambda_2}{4}$, follow the procedure discussed in class (Find $Z_2(-d)$).

$Z_2(-d) = \eta_2 \frac{\eta_3 + j\eta_2 \tan(\beta_2 d)}{\eta_2 + j\eta_3 \tan(\beta_2 d)}$

$d = \frac{\lambda_2}{4}; \beta_2 = \frac{2\pi}{\frac{\lambda}{4}}; \tan(\beta_2 d) = \tan\frac{\pi}{2} = \infty$

$\Rightarrow Z_2(-d) = \frac{\eta_2^2}{\eta_3}$, But $\eta_3 = \frac{\eta_0}{n_3} = 248\Omega; \eta_2 = \frac{377}{1.38} = 273$

$Z_2(-d) = 301\Omega = \eta_{eH}$ [for media 2 & 3]

$\Gamma_{eH} = \frac{\eta_{eH} - \eta_1}{\eta_{eH} + \eta_1} = \frac{301 - 377}{301 + 377} = \mathbf{-0.112 = \Gamma_{eH}}$

Problem 4 - 8.15

The magnetic field intensity in a medium is given by:
$\hat{H} = 0.1\mathrm{e}^{-77.485y} \cos\left(2\pi \times 10^9 t - 203.8y\right)\hat{a}_x$ A/m
If the medium is characterized by the free space permeability, determine the dielectric constant and the conductivity of the medium. Obtain the associated component of the $\bar{E}$ field. Compute the average power density.

Find σ & ϵ_r:
Comparing with the general equation for $\bar{H}$, get can extract the following terms:
$\alpha = 77.485; \beta = 203.8; \omega = 2\pi \times 10^9$ rad/s
We know, $\gamma^2 = (\alpha + \jmath\beta)^2 = \jmath\omega\mu\left(\sigma + \jmath\omega\epsilon\right) \Rightarrow \alpha^2 + \beta^2 + \jmath 2\alpha\beta = \jmath\omega\mu\sigma - \omega^2\mu\epsilon$
This yields two relationships, $2\alpha\beta = \omega\mu\sigma$ and $\omega^2\mu\epsilon = \beta^2 - \alpha^2$
Solve for σ: $\sigma = \frac{2\alpha\beta}{\omega\mu} = \frac{2*77.485*203.8}{2\pi\times10^9*4\pi\times10^{-7}} = \sigma = \mathbf{4\ S/m}$
$\epsilon_r = \frac{\beta^2-\alpha^2}{\omega^2\mu\epsilon_0} = \frac{(203.8)^2-(77.485)^2}{(2\pi\times10^9)^2*4\pi\times10^{-7}*8.85\times10^{-12}} = \mathbf{80.8} = \epsilon_\mathbf{r}$

Find $\bar{E}$ & P_{ave}
$\bar{E} = \eta\bar{H}; \eta$is complex
$\eta = \sqrt{\frac{\jmath\omega\mu}{\sigma+\jmath\omega\epsilon}} = |\eta|\,\mathrm{e}^{\jmath\theta_\eta}$
$|\eta| = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}}; \tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon}$
$\tan 2\theta_\eta = \frac{4}{2\pi\times10^9*80.8*8.85\times10^{-12}} = \frac{8}{9} = 0.88 \Rightarrow \theta_\eta = \mathbf{\frac{1}{2}}\tan^{-\mathbf{1}}\mathbf{0.88} = \mathbf{20.8}^\circ$
$|\eta| = \frac{\sqrt{\frac{4\pi\times10^{-7}}{80.8*8.85\times10^{-12}}}}{1.046} = \frac{377}{1.046\sqrt{80.8}} = 36.2\Omega$
$\hat{a}_k = \hat{a}_y; \hat{a}_H = \hat{a}_x$
$\hat{a}_E \times \hat{a}_H = \hat{a}_k; \hat{a}_E \times \hat{a}_x = \hat{a}_y \Rightarrow \hat{A}_E = \hat{a}_z$
$\bar{E} = \eta\bar{H}\hat{a}_E = 3.62\mathrm{e}^{-77485}\cos\left(2\pi \times 10^9 t - 203.8y + \theta_\eta\right)\hat{a}_z$ V/m
$P_{ave} = \frac{1}{2}\left[E \times H^*\right] = \frac{1}{2} * 3.62 * 0.1\mathrm{e}^{-2*77.485y}\cos 20.8^\circ$
$\mathbf{P_{ave}} = \mathbf{0.169}\mathrm{e}^{-\mathbf{154.97y}}\hat{a}_y\ \mathbf{W/m^2}$

Problem 5 - 8.17

A uniform plane wave is propagating in a good conductor. If the magnetic field intensity is given by:
$\hat{H} = 0.1\mathrm{e}^{-15z}\cos\left(2\pi \times 10^{8}t - 15z\right)\hat{a}_x$ A/m
determine the conductivity and the corresponding component of the $\bar{E}$ field. Calculate the average power loss in a block of unit area and δ thickness.

Find σ & δ

Comparing with the general equation for $\bar{H}$, get can extract the following terms:
$\alpha = 15; \beta = 15; \omega = 2\pi \times 10^{8}$ rad/s; $\alpha = \beta = \sqrt{\pi\mu f\sigma}$
$\sigma = \frac{\alpha^2}{\pi\mu f} = \frac{225}{\pi * 4\pi \times 10^{-7} * 10^{8}} = \mathbf{0.57\ S/m} = \sigma$
$\delta = \frac{1}{\alpha} = \frac{1}{15} = \mathbf{0.067m} = \delta$

Find $\bar{E}$ & Power loss for a unit area

Assume $\epsilon = \epsilon_0$
$\frac{\sigma}{\omega\epsilon} = \frac{0.57}{2\pi \times 10^{8} * 8.85 \times 10^{-12}} = \frac{57}{2\pi * 8.85} = 1.025$
$\theta_\eta = \frac{1}{2}\tan^{-1}(1.025) = 22.85°$
$|\eta| = \frac{377}{1.20} = 315\Omega$ (For more detail, see the solution for Problem 4 - 8.15)
$\mathbf{\bar{E}\left(z, t\right) = -31.5e^{-15z}\cos\left(2\pi \times 10^{8}t - 15z + \theta_\eta\right)\hat{a}_y\ V/m}$
$P_{ave} = \frac{1}{2}\left[E \times H^*\right] = \frac{1}{2} * 0.1 * 31.5\mathrm{e}^{-30z}\cos 22.85° = 1.45\mathrm{e}^{-30z}$ W/m^2
At $z = 0$, $P_{ave}(0) = 1.45$; At $z = \delta$, $P_{ave}(\delta) = 1.45\mathrm{e}^{-2} = 1.45 * 0.135$
Power Loss $= P_{ave}(0) - P_{ave}(\delta) = 1.45\left[1 - 0.135\right] * 1\mathrm{m}^2$
Power Loss = 1.254W

Problem 6 - 8.22

Find the polarization of the following waves:

a) $\bar{E} = 100\mathrm{e}^{-\jmath 300x}\hat{a}_y + 100\mathrm{e}^{-\jmath 300x}\hat{a}_z$ V/m
$\bar{E} = \bar{E}_y\hat{a}_y + \bar{E}_z\hat{a}_z$ (Notice E_y and E_z are in phase)
$\frac{\bar{E}_y}{\bar{E}_z} = \frac{100\cos(\omega t - 300x)}{100\cos(\omega t - 300x)} = 1 \Rightarrow$ **Linear Polarization**
$E(x,t) = [100\cos(\omega t - 300x)](\hat{a}_y + \hat{a}_z)$

b) $\bar{E} = 16\mathrm{e}^{\jmath\frac{\pi}{4}}\mathrm{e}^{-\jmath 100z}\hat{a}_x - 9\mathrm{e}^{-\jmath\frac{\pi}{4}}\mathrm{e}^{-\jmath 100z}\hat{a}_y$ V/m
$E(z,t) = 16\cos\left(\omega t - 100z + \frac{\pi}{4}\right)\hat{a}_x - 9\cos\left(\omega t - 100z - \frac{\pi}{4}\right)\hat{a}_y$
$E_x = 16\cos\left(\omega t - 100z + \frac{\pi}{4}\right)\hat{a}_x$
$E_y = -9\cos\left(\omega t - 100z - \frac{\pi}{4}\right)\hat{a}_y$

- Amplitudes of E_x and E_y are difference ($E_{mx} \neq E_{my}$)
- Phase difference $= \frac{\pi}{2} \Rightarrow$ **Elliptical Polarization**

c) $\bar{E} = 3\cos(t - 0.5y)\hat{a}_x - 4\sin(t - 0.5y)\hat{a}_z$ V/m
$E_x = 3\cos(t - 0.5y)\hat{a}_x$
$E_z = -4\sin(t - 0.5y)\hat{a}_z$

- Amplitudes of E_x and E_z are difference ($E_{mx} \neq E_{mz}$)
- Phase difference $= \frac{\pi}{2} \Rightarrow$ **Elliptical Polarization**