# Appendix B
# Generation of Random Numbers

We have seen that the Monte Carlo procedure is based on the generation of random numbers. Nowadays most computers contain routines that generate random numbers evenly distributed between 0 and 1.
In this Appendix we shall describe how random numbers are generated in general and we shall see how random numbers with any probability distribution can be obtained starting from a sequence of random numbers evenly distributed between 0 and 1. These numbers are designates in the present paper by the letter $r$.

## Generation of Evenly Distributed Random Numbers

The method of generation most commonly used in the computer routines for the generation of evenly distributed random numbers is the multiplicative congruential method: the $i$th element $r_i$ of the sequence is given by the previous element $r_{i-1}$ by a relation such as

$$r_i = pr_{i-1} \pmod q, \tag{B.1}$$

where $p$ and $q$ are appropriate constants. The first element of the sequence (seed) must be given by the user. The numbers $r_i$ of the sequence in Eq. (B.1) are obtained with a precise mathematical algorithm, and therefore they are not at all random; in fact, given the seed of the sequence, all its numbers are perfectly predictable. However, for "good" choices of the constants $p$ and $q$, the sequences of $r_i$ behave as random in the sense that they passed a large number of statistical tests of randomness. Numbers $r$ of such a type are called pseudorandom numbers. They have the advantage over truly random numbers of being generated in a fast way and of being reproducible, when desired, especially for program debugging.

## Generation of Random Numbers with Given Distributions

Random numbers $x$ with a given probability distribution $f(x)$ in an interval $(a, b)$ can be obtained starting from numbers $r$ evenly distributed in the interval (0, 1) with different techniques [1, 2]. We shall describe here the three simplest of them, which are usually used in Monte Carlo transport calculations.

## a) Direct Technique

If the function $f(x)$ is normalized to one in the interval of definition $(a, b)$, let us call $F(x)$ the integral function of $f$. Then, given a number $r$, we correspondingly choose $x_r$ such that

$$r = F(x_r) = \int_a^{x_r} f(x)\,dx. \tag{B.2}$$

The probability $P(x)\,dx$ that $x_r$ obtained in this way lies within an interval $dx$ around $x$ is equal to the corresponding $dF$, since $r$ has a flat distribution. Thus (see Fig. B.1)

$$P(x)\,dx = dF = f(x)\,dx, \tag{B.3}$$

as desired. If $f(x)$ is not normalized, then Eq. (B.2) must be replaced by

$$r = \int_a^{x_r} f(x)\,dx \bigg/ \int_a^b f(x)\,dx \tag{B.4}$$

with a constant $f(x)$, this technique provides the obvious formula for the generation of a random number evenly distributed between $a$ and $b$:

$$x_r = a + (b - a)r. \tag{B.5}$$

Often, in real cases, the above simple direct technique cannot be used because it is not possible either to evaluate analytically the integral in

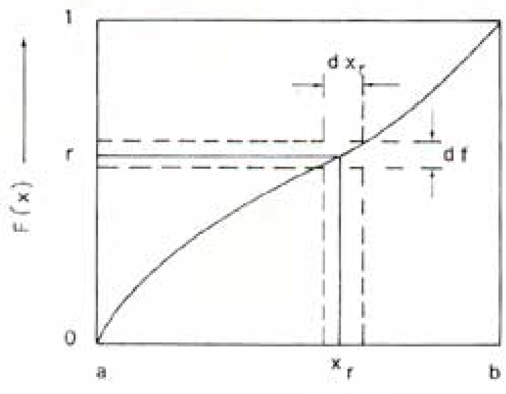


Fig. B.1. Direct technique for the generation of a random variable $x_r$ with probability function $f(x) = dF(x)/dx$ starting from random numbers $r$ evenly distributed between 0 and 1. (See text)

Eq. (B.2) or to solve with respect to $x$, the equation which results from Eq. (B.2). In these cases one of the following techniques can be used.

## b) Rejection Technique

Let C be a positive number such that

$$C \geqslant f(x) \tag{B.6}$$

in the whole interval $(a, b)$, and let $r_1$ and $r'_1$ be two random numbers obtained with a flat distribution in (0, 1). Then

$$x_1 = a + (b - a)r_1 \qquad \text{and} \qquad f_1 = r'_1 C \tag{B.7}$$

are two random numbers obtained with a flat distribution in $(a, b)$ and $(0, C)$, respectively. If

$$f_1 \leqslant f(x_1) \tag{B.8}$$

then $x_1$ is retained as choice of $x$, otherwise $x_1$ is rejected, and a new pair $r_2$, $r'_2$ is generated; the process is repeated until Eq. (B.8) is satisfied. Since for each pair $r_i$, $r'_i$ a point with coordinates $(x_i, f_i)$ is obtained from the uniform distribution in the rectangle $abCC$ of Fig. B.2, the probability that $x_i$ within an interval $dx$ around $x$ will be accepted is proportional to the probability that $x_i$ lies within this interval, proportional to $dx$, times the probability of accepting $x_i$, proportional to $f(x_i)$, as desired.
The above technique is always applicable, with any bounded $f(x)$ in a finite interval $(a, b)$. However, when $f(x)$ is strongly peaked, many pairs of numbers might be generated before a successful trial, with a resulting large

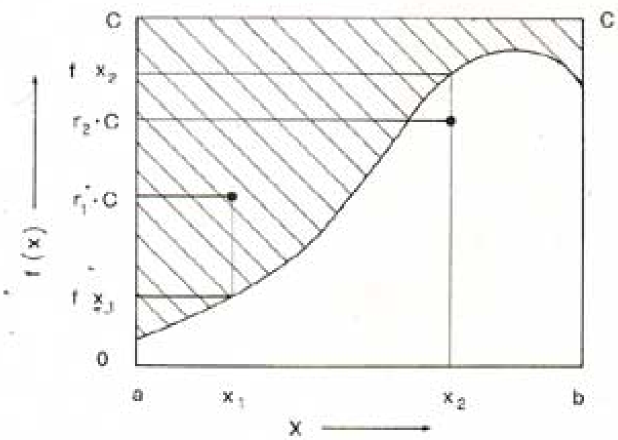


Fig. B.2. Rejection technique for the generation of a random variable $x$, with probability function $f(x)$ starting from random numbers $r$ evenly distributed between 0 and 1. (See text) expense of computer time. The technique described below may overcome this difficulty by combining the two previous techniques.

### c) Combined Technique

Let $x_1$ be a random number generated with the direct technique according to a distribution $g(x)$. If, furthermore, $K$ is a constant such that

$$Kg(x) \geqslant f(x) \tag{B.9}$$

in the whole range $(a, b)$ of interest, a new random number $r_1$ is generated in $(0, 1)$ and $x_1$ is accepted as a value of the random variable $x$ if

$$r_1 Kg(x_1) < f(x_1). \tag{B.10}$$

In fact, in this way the probability of having an accepted $x_1$ within an interval $dx$ around $x$ is proportional to the probability that $x_1$ lies within this interval, proportional to $g(x)\,dx$, times the probability of accepting $x_1$, proportional to the ratio $f(x)/Kg(x)$. The final probability is therefore proportional to $f(x)\,dx$, as desired. A geometrical interpretation of the combined technique is shown in Fig. B.3: the selection of $x_1$ is equivalent to the generation of a point with flat distribution below the curve $Kg(x)$, while the condition of acceptance requires that the point lies in the area below the curve $f(x)$. If the curve $Kg(x)$ is not too far from $f(x)$, few attempts will be necessary per successful trial.
The rejection technique described in the previous section is a particular case of the combined technique, for $g = \text{const}$. The combined technique, however, can also be applied to unbounded functions or to functions defined in unbounded intervals.

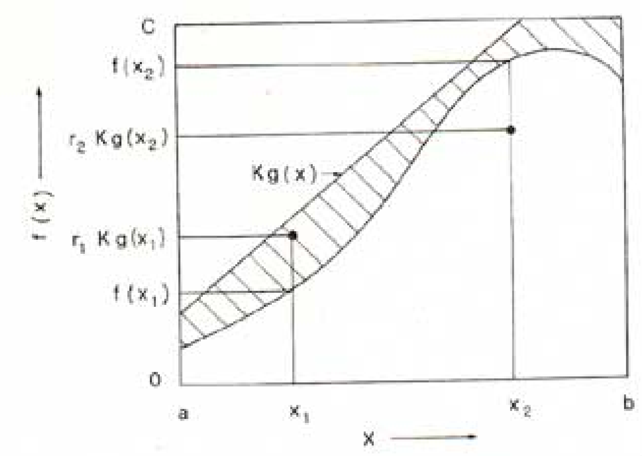


Fig. B.3. Combined technique for the generation of a random variable $x$, with probability function $f(x)$ starting from random numbers $r$ evenly distributed between 0 and 1. (See text)

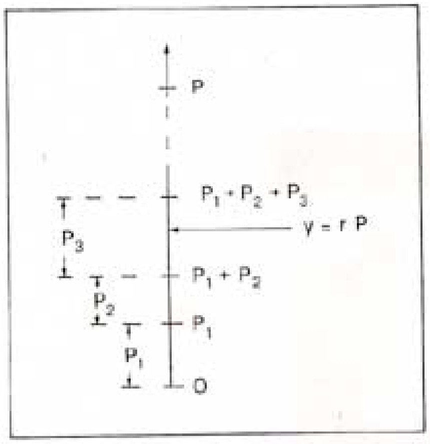


Fig. B.4. Generation of a random discrete variable $i$ with probability function $P_i$ starting from random numbers $r$ evenly distributed between 0 and 1. (See text)

As we said above, other techniques can be used for special cases. As an example we may mention the use of the sum of a certain number of random numbers $r$ to generate $x$ according to a normal distribution [3]. For such a distribution the combined technique can also be used [4].

d) Discrete Case

When an event must be chosen among a given number of different possibilities or, in other words, when the variable $x$ must be chosen from a discrete set, the direct technique can be used with $f(x)$ given by the sum of $\delta$ functions. Figure B.4 illustrates this case: if $P_i$ is the probability of occurrence of the $i$th event $x_i$, then a random number

$$y = rP, \tag{B.11}$$

where

$$P = \sum_i P_i \tag{B.12}$$

is generated and compared successively with $P_1, P_1 + P_2, P_1 + P_2 + P_3, \ldots$. The $j$th event is chosen if $j$ is such that the first of the above partial sums which is larger than $y$ is $P_1 + P_2 + \cdots + P_j$. Figure B.4 shows immediately that the probability of choosing the $j$th event is proportional to $P_j$, as desired.

## References

[1] Hammersley, J. M., Handscomb, D. C.: Monte Carlo Methods (Barlett, M. S., ed.). London: Methuen-Chapman and Hall. 1964.