Chapter 2

Brief Review of Electromagnetics

2.1 Introduction

The specific equations on which the finite-difference time-domain (FDTD) method is based will be considered in some detail later. The goal here is to remind you of the physical significance of the equations to which you have been exposed in previous courses on electromagnetics.

In some sense there are just a few simple premises which underlie all electromagnetics. One could argue that electromagnetics is simply based on the following:

1. Charge exerts force on other charge.
2. Charge in motion exerts a force on other charge in motion.
3. All material is made up of charged particles.

Of course translating these premises into a corresponding mathematical framework is not trivial. However one should not lose sight of the fact that the math is trying to describe principles that are conceptually rather simple.

2.2 Coulomb's Law and Electric Field

Coulomb studied the electric force on charged particles. As depicted in Fig. 2.1, given two discrete particles carrying charge Q_1 and Q_2, the force experienced by Q_2 due to Q_1 is along the line joining Q_1 and Q_2. The force is proportional to the charges and inversely proportional to the square of the distance between the charges. A proportionality constant is needed to obtain Coulomb's law which gives the equation of the force on Q_2 due to Q_1:

$$\mathbf{F}_{12} = \hat{\mathbf{a}}_{12} \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{R_{12}^2} \tag{2.1}$$

where $\hat{\mathbf{a}}_{12}$ is a unit vector pointing from Q_1 to Q_2, R_{12} is the distance between the charges, and $1/4\pi\epsilon_0$ is the proportionality constant. The constant ϵ_0 is known as the permittivity of free space

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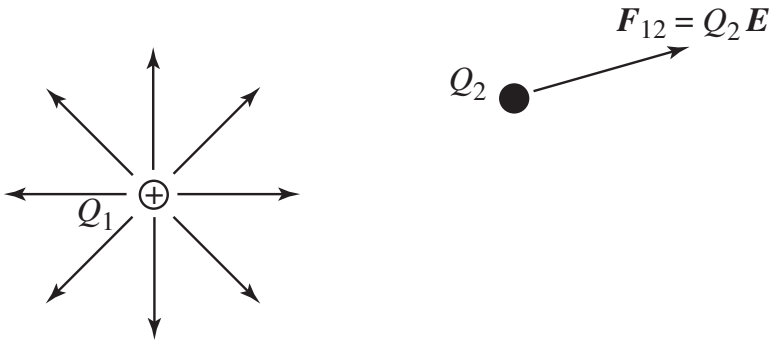

Figure 2.1: The force experienced by charge Q_2 due to charge Q_1 is along the line which pass through both charges. The direction of the force is dictate by the signs of the charges. Electric field is assumed to point radially away from positive charges as is indicated by the lines pointing away from Q_1 (which is assumed here to be positive).

and equals approximately 8.854×10^{-12} F/m. Charge is expressed in units of Coulombs (C) and can be either negative or positive. When the two charges have like signs, the force will be repulsive: $\mathbf{F}_{12}$ will be parallel to $\hat{\mathbf{a}}_{12}$. When the charges are of opposite sign, the force will be attractive so that $\mathbf{F}_{12}$ will be anti-parallel to $\hat{\mathbf{a}}_{12}$.

There is a shortcoming with (2.1) in that it implies action at a distance. It appears from this equation that the force $\mathbf{F}_{12}$ is established instantly. From this equation one could assume that a change in the distance R_{12} results in an instantaneous change in the force $\mathbf{F}_{12}$, but this is not the case. A finite amount of time is required to communicate the change in location of one charge to the other charge (similarly, it takes a finite amount of time to communicate a change in the quantity of one charge to the other charge). To overcome this shortcoming it is convenient to employ the concept of fields. Instead of Q_1 producing a force directly on Q_2, Q_1 is said to produce a field. This field then produces a force on Q_2. The field produced by Q_1 is independent of Q_2—it exists whether or not Q_2 is there to experience it.

In the static case, the field approach does not appear to have any advantage over the direct use of Coulomb's law. This is because for static charges Coulomb's law is correct. Fields must be time-varying for the distinction to arise. Nevertheless, to be consistent with the time-varying case, fields are used in the static case as well. The electric field produced by the point charge Q_1 is

$$\mathbf{E}_1 = \hat{\mathbf{a}}_r \frac{Q_1}{4\pi\epsilon_0 r^2} \tag{2.2}$$

where $\hat{\mathbf{a}}_r$ is a unit vector which points radially away from the charge and r is the distance from the charge. The electric field has units of volts per meter (V/m).

To find the force on Q_2, one merely takes the charge times the electric field: $\mathbf{F}_{12} = Q_2\mathbf{E}_1$. In general, the force on any charge Q is the product of the charge and the electric field at which the charge is present, i.e., $\mathbf{F} = Q\mathbf{E}$.

2.3 Electric Flux Density

All material is made up of charged particles. The material may be neutral overall because it has as many positive charges as negative charges. Nevertheless, there are various ways in which the positive and negative charges may shift slightly within the material, perhaps under the influence of an electric field. The resulting charge separation will have an effect on the overall electric field. Because of this it is often convenient to introduce a new field known as the electric flux density, $\mathbf{D}$, which has units of Coulombs per square meter (C/m^2).* Essentially the $\mathbf{D}$ field ignores the local effects of charge which is bound in a material.

In free space, the electric field and the electric flux density are related by

$$\mathbf{D} = \epsilon_0 \mathbf{E}. \tag{2.3}$$

Gauss's law states that integrating $\mathbf{D}$ over a closed surface yields the enclosed free charge

$$\oint_S \mathbf{D} \cdot \mathbf{ds} = Q_{\text{enc}} \tag{2.4}$$

where S is the closed surface, $\mathbf{ds}$ is an incremental surface element whose normal is directed radially outward, and Q_{enc} is the enclosed charge. As an example, consider the electric field given in (2.2). Taking S to be a spherical surface with the charge at the center, it is simple to perform the integral in (2.4):

$$\oint_S \mathbf{D} \cdot \mathbf{ds} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \epsilon_0 \frac{Q_1}{4\pi\epsilon_0 r^2} \hat{\mathbf{a}}_r \cdot \hat{\mathbf{a}}_r r^2 \sin\theta \, d\phi \, d\theta = Q_1. \tag{2.5}$$

The result is actually independent of the surface chosen (provided it encloses the charge), but the integral is especially easy to perform for a spherical surface.

We want the integral in (2.4) always to equal the enclosed charge as it does in free space. However, things are more complicated when material is present. Consider, as shown in Fig. 2.2, two large parallel plates which carry uniformly distributed charge of equal magnitude but opposite sign. The dashed line represents an integration surface S which is assumed to be sufficiently far from the edges of the plate so that the field is uniform over the top of S. This field is identified as $\mathbf{E}_0$. The fields are zero outside of the plates and are tangential to the sides of S within the plates. Therefore the only contribution to the integral would be from the top of S. The result of the integral $\oint_S \epsilon_0 \mathbf{E} \cdot \mathbf{ds}$ is the negative charge enclosed by the surface (i.e., the negative charge on the bottom plate which falls within S).

Now consider the same plates, carrying the same charge, but with a material present between the plates. Assume this material is "polarizable" such that the positive and negative charges can shift slightly. The charges are not completely free to move—they are bound charges. The positive charges will be repelled by the top plate and attracted to the bottom plate. Conversely, the negative charges will be repelled by the bottom plate and attracted to the top plate. This scenario is depicted in Fig. 2.3.

*Note that not everybody advocates using the $\mathbf{D}$ field. See for example Volume II of *The Feynman Lectures on Physics*, R. P. Feynman, R. B. Leighton, and M. Sands, Addison-Wesley, 1964. Feynman only uses $\mathbf{E}$ and never resorts to $\mathbf{D}$.

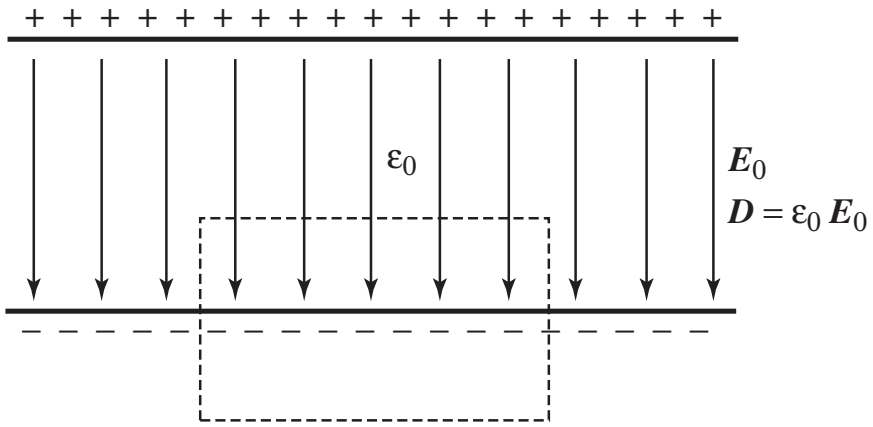

Figure 2.2: Charged parallel plates in free space. The dashed line represents the integration surface S.

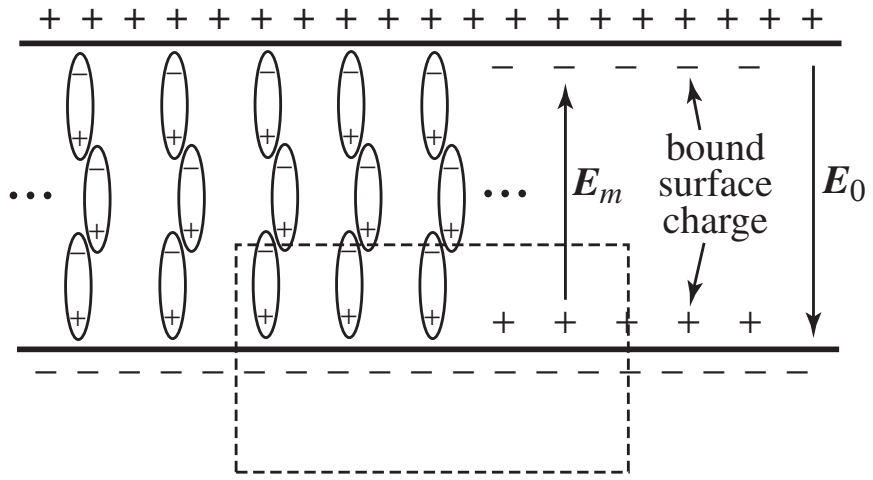

Figure 2.3: Charged parallel plates with a polarizable material present between the plates. The elongated objects represent molecules whose charge orientation serves to produce a net bound negative charge layer at the top plate and a bound positive charge layer at the bottom plate. In the interior, the positive and negative bound charges cancel each other. It is only at the surface of the material where one must account for the bound charge. Thus, the molecules are not drawn throughout the figure. Instead, as shown toward the right side of the figure, merely the bound charge layer is shown. The free charge on the plates creates the electric field $\mathbf{E}_0$. The bound charge creates the electric field $\mathbf{E}_m$ which opposes $\mathbf{E}_0$ and hence diminishes the total electric field. The dashed line again represents the integration surface S.

With the material present the electric field due to the charge on the plates is still $\mathbf{E}_0$, i.e., the same field as existed in Fig. (2.2). However, there is another field present due to the displacement of the bound charge in the polarizable material between the plates. The polarized material effectively acts to establish a layer of positive charge adjacent to the bottom plate and a layer of negative charge adjacent to the top plate. The field due to these layers of charge is also uniform but it is in the opposite direction of the field caused by the "free charge" on the plates. The field due to bound charge is labeled $\mathbf{E}_m$ in Fig. (2.3). The total field is the sum of the fields due to the bound and free charges, i.e., $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_m$. Because $\mathbf{E}_0$ and $\mathbf{E}_m$ are anti-parallel, the magnitude of the total electric field $\mathbf{E}$ will be less than $\mathbf{E}_0$.

Since the material is neutral, we would like the integral of the electric flux over the surface S to yield just the enclosed charge on the bottom plate—not the bound charge due to the material. In some sense this implies that the integration surface cannot separate the positive and negative bound charge of any single molecule. Each molecule is either entirely inside or outside the integration surface. Since each molecule is neutral, the only contribution to the integral will be from the free charge on the plate.

With the material present, the integral of $\oint_S \epsilon_0 \mathbf{E} \cdot \mathbf{ds}$ yields too little charge. This is because, as stated above, the total electric field $\mathbf{E}$ is less than it would be if only free space were present. To correct for the reduced field and to obtain the desired result, the electric flux density is redefined so that it accounts for the presence of the material. The more general expression for the electric flux density is

$$\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E} = \epsilon \mathbf{E} \tag{2.6}$$

where ϵ_r is the relative permittivity and ϵ is called simply the permittivity. By accounting for the permittivity of a material, Gauss's law is always satisfied.

In (2.6), $\mathbf{D}$ and $\mathbf{E}$ are related by a scalar constant. This implies that the $\mathbf{D}$ and $\mathbf{E}$ fields are related by a simple proportionality constant for all frequencies, all orientations, and all field strengths. Unfortunately the real world is not so simple. Clearly if the electric field is strong enough, it would be possible to tear apart the bound positive and negative charges. Since charges have some mass, they do not react the same way at all frequencies. Additionally, many materials may have some structure, such as crystals, where the response in one direction is not the same in other directions. Nevertheless, Gauss's law is the law and thus always holds. When things get more complicated one must abandon a simple scalar for the permittivity and use an appropriate form to ensure Gauss's law is satisfied. So, for example, it may be necessary to use a tensor for permittivity that is directionally dependent. However, with the exception of frequency-dependent behavior (i.e., dispersive materials), we will not be pursuing those complications. A scalar permittivity will suffice.

2.4 Static Electric Fields

Ignoring possible nonlinear behavior of material, superposition holds for electromagnetic fields. Therefore we can think of any distribution of charges as a collection of point charges. We can get the total field by summing the contributions from all the charges (and this summing will have to be in the form of an integration if the charge is continuously distributed).

Note from (2.2) that the field associated with a point charge merely points radially away from the charge. There is no "swirling" of the field. If we have more than a single charge, the total

field may bend, but it will not swirl. Imagine a tiny wheel with positive charge distributed around its circumference. The wheel hub of the wheel is held at a fixed location but the wheel is free to spin about its hub. For static electric fields, no matter where we put this wheel, there would be no net force on the wheel to cause it to spin. There may be a net force pushing the entire wheel in a particular direction (a translational force), but the forces which are pushing the wheel to spin in the clockwise direction are balanced by the forces pushing the wheel to spin in the counterclockwise direction.

Another property of electrostatic fields is that the electric flux density only begins or terminates on free charge. If there is no charge present, the lines of flux continue.

The lack of swirl in the electric field and the source of electric flux density are fairly simple concepts. However, to be able to analyze the fields properly, one needs a mathematical statement of these concepts. The appropriate statements are

$$\nabla \times \mathbf{E} = 0 \tag{2.7}$$

and

$$\nabla \cdot \mathbf{D} = \rho_v \tag{2.8}$$

where ∇ is the del or nabla operator and ρ_v is the electric charge density (with units of C/m^3). Equation (2.7) is the curl of the electric field and (2.8) is the divergence of the electric flux density. These two equations are discussed further in the following section.

2.5 Gradient, Divergence, and Curl

The del operator is independent of the coordinate system used—naturally the behavior of the fields should not depend on the coordinate system used to describe the field. Nevertheless, the del operator can be expressed in different coordinates systems. In Cartesian coordinates del is

$$\nabla \equiv \hat{\mathbf{a}}_x \frac{\partial}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial}{\partial z} \tag{2.9}$$

where the symbol $\equiv$ means "defined as."

Del acting on a scalar field produces the gradient of the field. Assuming f is a some scalar field, ∇f produces the vector field given by

$$\nabla f = \hat{\mathbf{a}}_x \frac{\partial f}{\partial x} + \hat{\mathbf{a}}_y \frac{\partial f}{\partial y} + \hat{\mathbf{a}}_z \frac{\partial f}{\partial z}. \tag{2.10}$$

The gradient of f points in the direction of greatest change and is proportional to the rate of change. Assume we wish to find the amount of change in f for a small movement dx in the x direction. This can be obtained via $\nabla f \cdot \hat{\mathbf{a}}_x dx$, to wit

$$\nabla f \cdot \hat{\mathbf{a}}_x dx = \frac{\partial f}{\partial x} dx = (\text{rate of change in } x \text{ direction}) \times (\text{movement in } x \text{ direction}). \tag{2.11}$$

This can be generalized for movement in an arbitrary direction. Letting an incremental small length be given by

$$\mathbf{d}\ell = \hat{\mathbf{a}}_x dx + \hat{\mathbf{a}}_y dy + \hat{\mathbf{a}}_z dz, \tag{2.12}$$

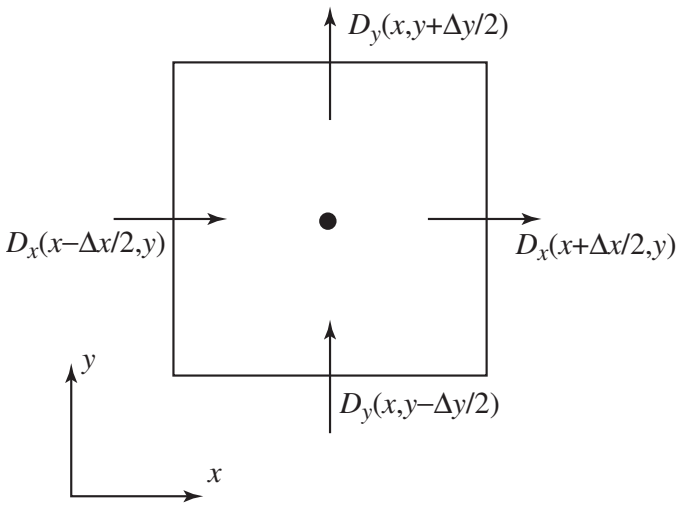

Figure 2.4: Discrete approximation to the divergence taken in the xy-plane.

the change in the field realized by moving an amount $\mathbf{d}\ell$ is

$$\nabla f \cdot \mathbf{d}\ell = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz. \tag{2.13}$$

Returning to (2.8), when the del operator is dotted with a vector field, one obtains the divergence of that field. Divergence can be thought of as a measure of "source" or "sink" strength of the field at a given point. The divergence of a vector field is a scalar field given by

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}. \tag{2.14}$$

Let us consider a finite-difference approximation of this divergence in the xy-plane as shown in Fig. 2.4. Here the divergence is measured over a small box where the field is assumed to be constant over each edge of the box. The derivatives can be approximated by central differences:

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} \approx \frac{D_x\left(x+\frac{\Delta_x}{2},y\right) - D_x\left(x-\frac{\Delta_x}{2},y\right)}{\Delta_x} + \frac{D_y\left(x,y+\frac{\Delta_y}{2}\right) - D_y\left(x,y-\frac{\Delta_y}{2}\right)}{\Delta_y} \tag{2.15}$$

where this is exact as Δ_x and Δ_y go to zero. Letting $\Delta_x = \Delta_y = \delta$, (2.15) can be written

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} \approx \frac{1}{\delta}\left(D_x\left(x+\frac{\delta}{2},y\right) - D_x\left(x-\frac{\delta}{2},y\right) + D_y\left(x,y+\frac{\delta}{2}\right) - D_y\left(x,y-\frac{\delta}{2}\right)\right). \tag{2.16}$$

Inspection of (2.16) reveals that the divergence is essentially a sum of the field over the faces with the appropriate sign changes. Positive signs are used if the field is assumed to point out of the box and negative signs are used when the field is assumed to point into the box. If the sum of these values is positive, that implies there is more flux out of the box than into it. Conversely, if the sum is negative, that means more flux is flowing into the box than out. If the sum is zero, there must

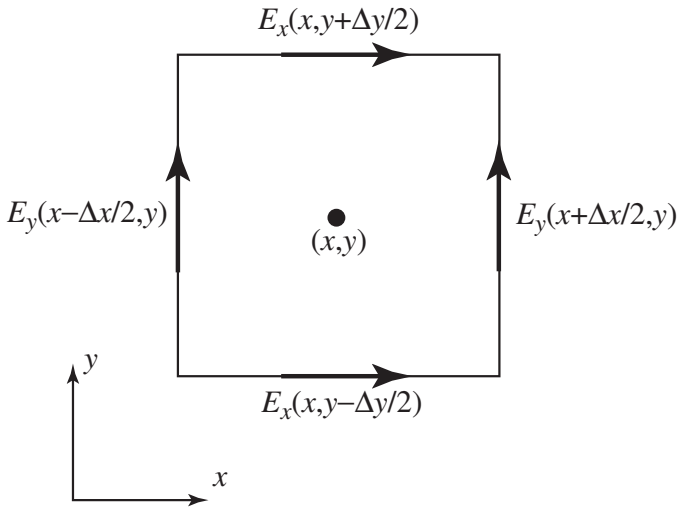

Figure 2.5: Discrete approximation to the curl taken in the xy-plane.

be as much flux flowing into the box as out of it (that does not imply necessarily that, for instance, $D_x\left(x+\delta/2,y\right)$ is equal to $D_x\left(x-\delta/2,y\right)$, but rather that the sum of all four fluxes must be zero).

Equation (2.8) tells us that the electric flux density has zero divergence except where there is charge present (as specified by the charge-density term ρ_v). If the charge density is zero, the total flux entering some small enclosure must also leave it. If the charge density is positive at some point, more flux will leave a small enclosure surrounding that point than will enter it. On the other hand, if the charge density is negative, more flux will enter the enclosure surrounding that point than will leave it.

Finally, let us consider (2.7) which is the curl of the electric field. In Cartesian coordinates it is possible to treat this operation as simply the cross product between the vector operator ∇ and the vector field $\mathbf{E}$:

$$\nabla\times\mathbf{E}=\begin{vmatrix}\hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=\hat{\mathbf{a}}_x\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\hat{\mathbf{a}}_y\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{\mathbf{a}}_z\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right). \tag{2.17}$$

Let us consider the behavior of only the z component of this operator which is dictated by the field in the xy-plane as shown in Fig. 2.5. The z-component of $\nabla\times\mathbf{E}$ can be written as

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\approx\frac{E_y\left(x+\frac{\Delta_x}{2},y\right)-E_y\left(x-\frac{\Delta_x}{2},y\right)}{\Delta_x}-\frac{E_x\left(x,y+\frac{\Delta_y}{2}\right)-E_x\left(x,y-\frac{\Delta_y}{2}\right)}{\Delta_y}. \tag{2.18}$$

The finite-difference approximations of the derivatives are again based on the fields on the edges of a box surrounding the point of interest. However, in this case the relevant fields are tangential to the edges rather than normal to them. Again letting $\Delta_x=\Delta_y=\delta$, (2.18) can be written

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\approx\frac{1}{\delta}\left(E_y\left(x+\frac{\delta}{2},y\right)-E_y\left(x-\frac{\delta}{2},y\right)-E_x\left(x,y+\frac{\delta}{2}\right)+E_x\left(x,y-\frac{\delta}{2}\right)\right). \tag{2.19}$$

In the sum on the right side the sign is positive if the vector component points in the counter-clockwise direction (relative to rotations about the center of the box) and is negative if the vector points in the clockwise direction. Thus, if the sum of these vector components is positive, that implies that the net effect of these electric field vectors is to tend to push a positive charge in the counterclockwise direction. If the sum were negative, the vectors would tend to push a positive charge in the clockwise direction. If the sum is zero, there is no tendency to push a positive charge around the center of the square (which is not to say there would not be a translation force on the charge—indeed, if the electric field is non-zero, there has to be some force on the charge).

2.6 Laplacian

In addition to the gradient, divergence, and curl, there is one more vector operator to consider. There is a vector identity that the curl of the gradient of any function is identically zero

$$\nabla \times \nabla f = 0. \tag{2.20}$$

This is simple to prove by merely performing the operations in Cartesian coordinates. One obtains several second-order partial derivatives which cancel if the order of differentiation is switched. Recall that for a static distribution of charges, $\nabla \times \mathbf{E} = 0$. Since the curl of the electric field is zero, it should be possible to represent the electric field as the gradient of some scalar function

$$\mathbf{E} = -\nabla V. \tag{2.21}$$

The scalar function V is the electric potential and the negative sign is used to make the electric field point from higher potential to lower potential (by historic convention the electric field points away from positive charge and toward negative charge). By expressing the electric field this way, the curl of the electric field is guaranteed to be zero.

Another way to express the relationship between the electric field and the potential is via integration. Consider movement from an arbitrary point a to an arbitrary point b. The change in potential between these two points can be expressed as

$$V_b - V_a = \int_a^b \nabla V \cdot \mathbf{d}\ell. \tag{2.22}$$

The integrand represent the change in the potential for a movement $\mathbf{d}\ell$ and the integral merely sums the changes over the path from a to b. However, the change in potential must also be commensurate with the movement in the direction of, or against, the electric field. If we move against the electric field, potential should go up. If we move along the electric field, the potential should go down. In other words, the incremental change in potential for a movement $\mathbf{d}\ell$ should be $dV = -\mathbf{E} \cdot \mathbf{d}\ell$ (if the movement $\mathbf{d}\ell$ is orthogonal to the electric field, there should be no change in the potential). Summing change in potential over the entire path yields

$$V_b - V_a = -\int_a^b \mathbf{E} \cdot \mathbf{d}\ell. \tag{2.23}$$

The integrals in (2.22) and (2.23) can be equated. Since the equality holds for any two arbitrary points, the integrands must be equal and we are again left with $\mathbf{E} = -\nabla V$.

The electric flux density can be related to the electric field via $\mathbf{D} = \epsilon\mathbf{E}$ and the behavior of the flux density $\mathbf{D}$ is dictated by $\nabla \cdot \mathbf{D} = \rho_v$. Combining these with (2.21) yields

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = -\nabla V. \tag{2.24}$$

Taking the divergence of both sides yields

$$\frac{1}{\epsilon}\nabla \cdot \mathbf{D} = \frac{1}{\epsilon}\rho_v = -\nabla \cdot \nabla V. \tag{2.25}$$

Rearranging this yields Poisson's equation given by

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \tag{2.26}$$

where ∇^2 is the Laplacian operator

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2.27}$$

Note that the Laplacian is a scalar operator. It can act on a scalar field (such as the potential V as shown above) or it can act on a vector field as we will see later. When it acts on a vector field, the Laplacian acts on each component of the field.

In the case of zero charge density, (2.26) reduces to Laplace's equation

$$\nabla^2 V = 0. \tag{2.28}$$

We have a physical intuition about what gradient, divergence, and curl are telling us, but what about the Laplacian? To answer this, consider a function of a single variable.

Given the function $V(x)$, we can ask if the function at some point is greater than, equal to, or less than the average of its neighboring values. The answer can be expressed in terms of the value of the function at the point of interest and the average of samples to either side of that central point:

$$\frac{V(x+\delta)+V(x-\delta)}{2} - V(x) = \begin{cases} \text{positive} & \text{if center point less than average of neighbors} \\ \text{zero} & \text{if center point equals average of neighbors} \\ \text{negative} & \text{if center point greater than average of neighbors} \end{cases} \tag{2.29}$$

Here the left-most term represents the average of the neighboring values and δ is some displacement from the central point. Equation (2.29) can be normalized by $\delta^2/2$ without changing the basic tenants of this equation. Performing that normalization and rearranging yields

$$\begin{aligned} \frac{1}{\delta^2/2}\left\{\frac{V(x+\delta)+V(x-\delta)}{2} - V(x)\right\} &= \frac{1}{\delta^2}\left\{(V(x+\delta)-V(x)) - (V(x)-V(x-\delta))\right\} \\ &= \frac{\frac{V(x+\delta)-V(x)}{\delta} - \frac{V(x)-V(x-\delta)}{\delta}}{\delta} \\ &\approx \frac{\frac{\partial V(x+\delta/2)}{\partial x} - \frac{\partial V(x-\delta/2)}{\partial x}}{\delta} \\ &\approx \frac{\partial^2 V(x)}{\partial x^2}. \end{aligned} \tag{2.30}$$

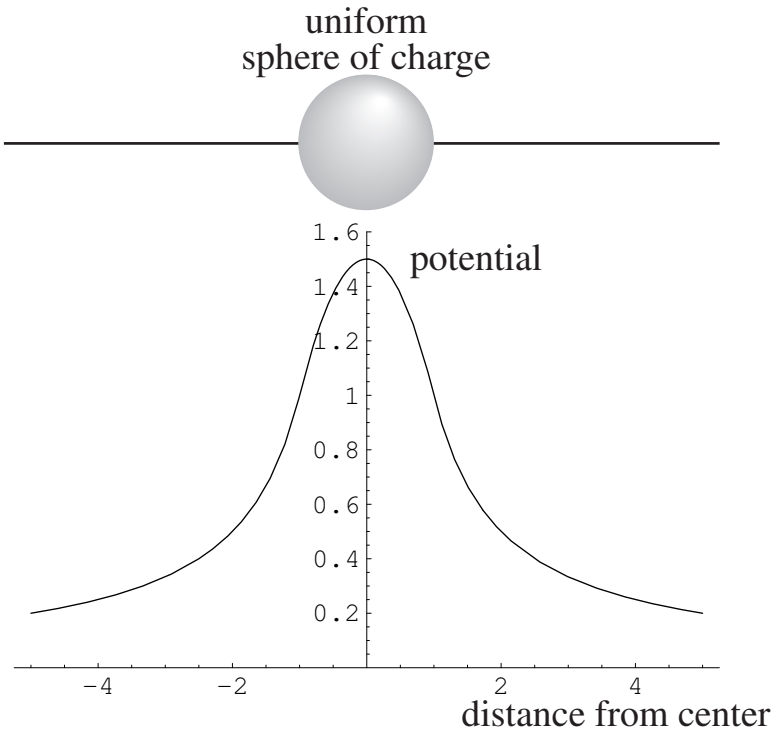

Figure 2.6: Potential along a path which passes through a uniform sphere of positive charge (arbitrary units).

Thus the second partial derivative can be thought of as a way of measuring the field at a point relative to its neighboring points. You should already have in mind that if the second derivative is negative, a function is tending to curve downward. Second derivatives are usually discussed in the context of curvature. However, you should also think in terms of the field at a point and its neighbors. At points where the second derivative is negative those points are higher than the average of their neighboring points (at least if the neighbors are taken to be an infinitesimally small distance away).

In lieu of these arguments, Poisson's equation (2.26) should have physical significance. Where the charge density is zero, the potential cannot have a local minima or maxima. The potential is always equal to the average of the neighboring points. If one neighbor is higher, the other must be lower (and this concept easily generalizes to higher dimensions). Conversely, if the charge density is positive over some region, the potential should increase as one moves deeper into that region but the rate of increase must be such that at any point the average of the neighbors is less than the center point. This behavior is illustrated in Fig. 2.6 which depicts the potential along a path through the center of a uniform sphere of charge.

2.7 Gauss's and Stokes' Theorems

Equation (2.4) presented Gauss's law which stated the flux of $\mathbf{D}$ through a closed surface S is equal to the enclosed charge. There is an identity in vector calculus, known as Gauss's theorem, which states that the integral of the flux of any vector field through a closed surface equals the integral of the divergence of the field over the volume enclosed by the surface. This holds for any vector field, but using the $\mathbf{D}$ field, Gauss's theorem states

$$\oint_S \mathbf{D} \cdot \mathbf{ds} = \int_V \nabla \cdot \mathbf{D} dv \tag{2.31}$$

where V is the enclosed volume and dv is a differential volume element. Note that the left-hand side of (2.31) is the left-hand side of (2.4).

The right-hand side of (2.4) is the enclosed charge Q_{enc} which could be determined either by evaluating the left-hand side of (2.4) or by integration of the charge density ρ_v over the volume enclosed by S. (This is similar to determining the mass of an object by integrating its mass density over its volume.) Thus,

$$Q_{\text{enc}} = \int_V \rho_v dv. \tag{2.32}$$

Equating the right-hand sides of (2.31) and (2.32) yields

$$\int_V \nabla \cdot \mathbf{D} dv = \int_V \rho_v dv. \tag{2.33}$$

Since this must hold over an arbitrary volume, the integrands must be equal which yields (2.8).

Another useful identity from vector calculus is Stokes' theorem which states that the integral of a vector field over any closed path is equal to the integral of the curl of that field over a surface which has that path as its border. Again, this holds for any vector field, but using the electric field as an example one can write

$$\oint_L \mathbf{E} \cdot \mathbf{d}\ell = \int_S \nabla \times \mathbf{E} \cdot \mathbf{ds}. \tag{2.34}$$

The surface normal is assumed to follow the right-hand convention so that when the fingers of the right hand are oriented along the path of the loop, the thumb points in the positive direction of the surface normal.

Static electric fields are conservative which means that the net work required to move a charge in a closed path is always zero. Along some portion of the path positive work would have to be done to push the charge against the field, but this amount of work would be given back by the field as the charge travels along the remaining portions of the path. The integrand on the left-hand side in (2.34) is the field dotted with an incremental length. If the integrand were multiplied by a unit positive charge, the integrand would represent work, since charge times field is force and force times distance is work. Because the electric field is conservative, the integral on the left-hand side of (2.34) must be zero. Naturally this implies that the integral on the right-hand side must also be zero. Since this holds for any loop L (or, similarly, any surface S), the integrand itself must be zero. Equating the integrand to zero yields (2.7).

2.8 Electric Field Boundary Conditions

Consider an interface between two homogeneous regions. Because electric flux density only begins or ends on charge, the normal component of $\mathbf{D}$ can only change at the interface if there is charge on the interface, i.e., surface charge is present. This can be stated mathematically as

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \tag{2.35}$$

where ρ_s is a surface charge density (C/m^2), $\hat{\mathbf{n}}$ is a unit vector normal to the surface, and $\mathbf{D}_1$ and $\mathbf{D}_2$ are the field to either side of the interface. One should properly argue this boundary condition by an application of Gauss's law for a small volume surrounding the surface, but such details are left to other classes (this is just a review!). If no charge is present, the normal components must be equal

$$\hat{\mathbf{n}} \cdot \mathbf{D}_1 = \hat{\mathbf{n}} \cdot \mathbf{D}_2. \tag{2.36}$$

The boundary conditions on the tangential component of the electric field can be determined by integrating the electric field over a closed loop which is essentially a rectangle which encloses a portion of the interface. By letting the sides shrink to zero and keeping the "top" and "bottom" of the rectangle small but finite (so that they are tangential to the surface), one essentially has that the field over the top must be the same as the field over the bottom (owing to the fact that total integral must be zero since the field is conservative). Stated mathematical, the boundary condition is

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0. \tag{2.37}$$

2.9 Conductivity and Perfect Electric Conductors

It is possible for the charge in materials to move under the influence of an electric field such that currents flow. If the material has a non-zero conductivity σ, the current density is given by

$$\mathbf{J} = \sigma \mathbf{E}. \tag{2.38}$$

The current density has units of A/m^2 and the conductivity has units of S/m.

If charge is building up or decaying in a particular region, the divergence of the current density must be non-zero. If the divergence is zero, that implies as much current leaves a point as enters it and there is no build-up or decay of charge. This can be stated as

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}. \tag{2.39}$$

If the divergence is positive, the charge density must be decreasing with time (so the negative sign will bring the two into agreement). This equation is a statement of charge conservation.

Perfect electric conductors (PECs) are materials where it is assumed that the conductivity approaches infinity. If the fields were non-zero in a PEC, that would imply the current was infinite. Since infinite currents are not allowed, the fields inside a PEC are required to be zero. This subsequently requires that the tangential electric field at the surface of a PEC is zero (since tangential fields are continuous across an interface and the fields inside the PEC are zero). Correspondingly, the normal component of the electric flux density $\mathbf{D}$ at the surface of a PEC must equal the charge density at the surface of the PEC. Since the fields inside a PEC are zero, all points of the PEC must be at the same potential.

2.10 Magnetic Fields

Magnetic fields circulate around, but they do not terminate on anything—there is no (known) magnetic charge. Nevertheless, it is often convenient to define magnetic charge and magnetic current. These fictions allow one to simplify various problems such as integral formulations of scattering problems. However for now we will stick to reality and say they do not exist.

The magnetic flux density $\mathbf{B}$ is somewhat akin to the electric field in that the force on a charge in motion is related to $\mathbf{B}$. If a charge Q is moving with velocity $\mathbf{u}$ in a field $\mathbf{B}$, it experiences a force

$$\mathbf{F} = Q\mathbf{u} \times \mathbf{B}. \tag{2.40}$$

Because $\mathbf{B}$ determines the force on a charge, it must account for all sources of magnetic field. When material is present, the charge in the material can have motion (or rotation) which influences the magnetic flux density.

Alternatively, similar to the electric flux density, we define the magnetic field $\mathbf{H}$ which ignores the local effects of material. These fields are related by

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H} = \mu \mathbf{H} \tag{2.41}$$

where μ_r is the relatively permeability, μ_0 is the permeability of free space equal to $4\pi \times 10^{-7}$ H/m, and μ is simply the permeability. Typically the relative permeability is greater than unity (although usually only by a small amount) which implies that when a material is present the magnetic flux density is larger than when there is only free space.

Charge in motion is the source of magnetic fields. If a current I flows over an incremental distance $\mathbf{d}\ell$, it will produce a magnetic field given by:

$$\mathbf{H} = \frac{I\mathbf{d}\ell \times \mathbf{a}_r}{4\pi r^2} \tag{2.42}$$

where $\mathbf{a}_r$ points from the location of the filament of current to the observation point and r is the distance between the filament and the observation point. Equation (2.42) is known as the Biot-Savart equation. Of course, because of the conservation of charge, a current cannot flow over just a filament and then disappear. It must flow along some path. Thus, the magnetic field due to a loop of current would be given by

$$\mathbf{H} = \oint_L \frac{I\mathbf{d}\ell \times \mathbf{a}_r}{4\pi r^2}. \tag{2.43}$$

If the current was flowing throughout a volume or over a surface, the integral would be correspondingly changed to account for the current wherever it flowed.

From (2.43) one sees that currents (which are just another way of saying charge in motion) are the source of magnetic fields. Because of the cross-product in (2.42) and (2.43), the magnetic field essentially swirls around the current. If one integrates the magnetic field over a closed path, the result is the current enclosed by that path

$$\oint_L \mathbf{H} \cdot \mathbf{d}\ell = I_{\text{enc}}. \tag{2.44}$$

The enclosed current I_{enc} is the current that passes through the surface S which is bound by the loop L.

The left-hand side of (2.44) can be converted to a surface integral by employing Stokes' theorem while the right-hand side can be related to the current density by integrating over the surface of the loop. Thus,

$$\oint_L \mathbf{H} \cdot \mathbf{d}\ell = \int_S \nabla \times \mathbf{H} \cdot \mathbf{ds} = I_{\text{enc}} = \int_S \mathbf{J} \cdot \mathbf{ds}. \tag{2.45}$$

Since this must be true for every loop (and surface), the integrands of the second and fourth terms can be equated. This yields

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{2.46}$$

The last equation needed to characterize static fields is

$$\nabla \cdot \mathbf{B} = 0. \tag{2.47}$$

This is the mathematical equivalent of saying there is no magnetic charge.

2.11 Magnetic Field Boundary Conditions

Note that the equation governing $\mathbf{B}$ is similar to the equation which governed $\mathbf{D}$. In fact, since the right-hand side is always zero, the equation for $\mathbf{B}$ is simpler. The arguments used to obtain the boundary condition for the normal component of the $\mathbf{D}$ field can be applied directly to the $\mathbf{B}$ field. Thus,

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0. \tag{2.48}$$

For the magnetic field, an integration path is constructed along the same lines as the one used to determine the boundary condition on the electric field. Note that the equations governing $\mathbf{E}$ and $\mathbf{H}$ are similar except that the one for $\mathbf{H}$ has a non-zero right-hand side. If the current density is zero over the region of interest, then there is really no distinction between the two and one can say that the tangential magnetic fields must be equal across a boundary. However, if a surface current exists on the interface, there may be a discontinuity in the tangential fields. The boundary condition is given by

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K} \tag{2.49}$$

where $\mathbf{K}$ is the surface current density (with units of A/m).

2.12 Summary of Static Fields

When a system is not changing with respect to time, the governing equations are

$$\nabla \cdot \mathbf{D} = \rho_v, \tag{2.50}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.51}$$

$$\nabla \times \mathbf{E} = 0, \tag{2.52}$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{2.53}$$

If a loop carries a current but is otherwise neutral, it will produce a magnetic field and only a magnetic field. If a charge is stationary, it will produce an electric field and only an electric field. The charge will not "feel" the loop current and the current loop will not feel the stationary charge (at least approximately). The magnetic field and electric field are decoupled. If a charge Q moves with velocity $\mathbf{u}$ in the presence of both an electric field and a magnetic field, the force on the charge is the sum of the forces due to the electric and magnetic fields

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{2.54}$$

2.13 Time Varying Fields

What happens when a point charge moves? We know that charge in motion gives rise to a magnetic field, but if the charge is moving, its associated electric field must also be changing. Thus, when a system is time-varying the electric and magnetic fields must be coupled.

There is a vector identity that the divergence of the curl of any vector field is identically zero. Taking the divergence of both sides of (2.53) yields

$$\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{2.55}$$

where the conservation of charge equation was used to write the last equality. Since the first term must be zero, this implies that $\partial \rho_v / \partial t$ must also be zero. However, that is overly restrictive. In general, for a time-varying system, the charge density will change with respect to time. Therefore something must be wrong with (2.53) as it pertains to time-varying fields. It was Maxwell who recognized that by adding the temporal derivative of the electric flux density to the right-hand side of (2.53) the equation would still be valid for the time-varying case. The correct equation is given by

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.56}$$

The term $\partial \mathbf{D} / \partial t$ is known as the displacement current while $\mathbf{J}$ is typically called the conduction current. Equation (2.56) is known as Ampere's law.

Taking the divergence of the right-hand side of (2.56) yields

$$\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = -\frac{\partial \rho_v}{\partial t} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = -\frac{\partial \rho_v}{\partial t} + \frac{\partial \rho_v}{\partial t} = 0 \tag{2.57}$$

where use was made of (2.8) and the conservation of charge equation (2.39).

The electromotive force (EMF) is the change in potential over some path. It has been observed experimentally that when a magnetic field is time-varying there is a non-zero EMF over a closed path which encloses the varying field (i.e., the electric field is no longer conservative).

The symbol λ is often use to represent total magnetic flux through a given surface, i.e.,

$$\lambda = \int_S \mathbf{B} \cdot \mathbf{ds}. \tag{2.58}$$

For time-varying fields, the EMF over a closed path L can be written

$$V_{\text{emf}} = \frac{d\lambda}{dt}, \tag{2.59}$$

$$-\oint_L \mathbf{E}\cdot\mathbf{d}\ell = \frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{ds}, \tag{2.60}$$

$$-\int_S \nabla\times\mathbf{E}\cdot\mathbf{ds} = \int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{ds}, \tag{2.61}$$

where Stokes' theorem was used to write the last equation. Since this equality holds over any surface, the integrands must be equal. This yields

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{2.62}$$

which is known as Faraday's law.

2.14 Summary of Time-Varying Fields

When a system is changing with respect to time, the governing equations are

$$\nabla\cdot\mathbf{D} = \rho_v, \tag{2.63}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{2.64}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{2.65}$$

$$\nabla\times\mathbf{H} = \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}. \tag{2.66}$$

Note that the divergence equations are unchanged from the static case. The two curl equations have picked up terms which couple the electric and magnetic fields. Since the additional terms both involve temporal derivatives, they go to zero in the static case and the equations reduce to those which governed static fields.

For time-varying fields the same boundary conditions hold as in the static case.

2.15 Wave Equation in a Source-Free Region

Equations (2.63)–(2.66) provide a set of coupled differential equations. In the FDTD method we will be dealing directly with the two curl equations. We will stick to the coupled equations and solve them directly. However, it is also possible to decouple these equations. As an example, taking the curl of both sides of (2.65) yields

$$\nabla\times\nabla\times\mathbf{E} = -\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\mu\nabla\times\frac{\partial\mathbf{H}}{\partial t}. \tag{2.67}$$

There is a vector identity that the curl of the curl of any field is given by

$$\nabla\times\nabla\times\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}. \tag{2.68}$$

(This is true for any field, not just the electric field as shown here.) In a source-free region, there is no free charge present, $\rho_v = 0$, and hence the divergence of the electric field is zero ($\nabla \cdot \mathbf{D} = \epsilon \nabla \cdot \mathbf{E} = 0$). Thus (2.67) can be written

$$\nabla^2 \mathbf{E} = \mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}). \tag{2.69}$$

Keeping in mind that we are considering a source-free region so that J would be zero, we now use (2.66) to write

$$\begin{aligned} \nabla^2 \mathbf{E} &= \mu \frac{\partial}{\partial t} \frac{\partial \mathbf{D}}{\partial t}, & (2.70) \\ &= \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}. & (2.71) \end{aligned}$$

Equation (2.71) is the wave equation for the electric field and is often written

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{2.72}$$

where $c = 1/\sqrt{\mu\epsilon}$. Had we decoupled the equations to solve $\mathbf{H}$ instead of $\mathbf{E}$, we would still obtain the same equation (except with $\mathbf{H}$ replacing $\mathbf{E}$).

2.16 One-Dimensional Solutions to the Wave Equation

The wave equation which governs either the electric or magnetic field in one dimension in a source-free region can be written

$$\frac{\partial^2 f(x,t)}{\partial x^2} - \mu\epsilon \frac{\partial^2 f(x,t)}{\partial t^2} = 0. \tag{2.73}$$

We make the claim that any function $f(\xi)$ is a solution to this equation provided that f is twice differentiable and ξ is replaced with $t \pm x/c$ where $c = 1/\sqrt{\mu\epsilon}$. Thus, we have

$$f(\xi) = f(t \pm x/c) = f(x,t). \tag{2.74}$$

The first derivatives of this function can be obtained via the chain rule. Keeping in mind that

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= 1, & (2.75) \\ \frac{\partial \xi}{\partial x} &= \pm \frac{1}{c}, & (2.76) \end{aligned}$$

the first derivatives can be written

$$\begin{aligned} \frac{\partial f(\xi)}{\partial t} &= \frac{\partial f(\xi)}{\partial \xi} \frac{\partial \xi}{\partial t} = \frac{\partial f(\xi)}{\partial \xi}, & (2.77) \\ \frac{\partial f(\xi)}{\partial x} &= \frac{\partial f(\xi)}{\partial \xi} \frac{\partial \xi}{\partial x} = \pm \frac{1}{c} \frac{\partial f(\xi)}{\partial \xi}. & (2.78) \end{aligned}$$

Employing the chain rule in a similar fashion, the second derivatives can be written as

$$\frac{\partial}{\partial t}\left(\frac{\partial f(\xi)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial f(\xi)}{\partial \xi}\right) = \frac{\partial^2 f(\xi)}{\partial \xi^2}\frac{\partial \xi}{\partial t} = \frac{\partial^2 f(\xi)}{\partial \xi^2}, \tag{2.79}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f(\xi)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\pm\frac{1}{c}\frac{\partial f(\xi)}{\partial \xi}\right) = \pm\frac{1}{c}\frac{\partial^2 f(\xi)}{\partial \xi^2}\frac{\partial \xi}{\partial x} = \frac{1}{c^2}\frac{\partial^2 f(\xi)}{\partial \xi^2}. \tag{2.80}$$

Thus, (2.79) and (2.80) show that

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial \xi^2} \tag{2.81}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial \xi^2}. \tag{2.82}$$

Substituting these into (2.73) yields

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial \xi^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial \xi^2} = 0. \tag{2.83}$$

The two terms on the left-hand side cancel, thus satisfying the equation.

Consider a constant argument of f, say $t - x/c = 0$. Assume this argument is obtain by simultaneously having $t = 0$ and $x = 0$. Now, let time advance by one second, i.e., $t = 1$ s. How must the position x change to maintain an argument of zero? Solving for x yields $x = c(1\text{ s})$. In other words to move along with the field so as to maintain the value $f(0)$ (whatever that value happens to be), at time zero, we would be at position zero. At time one second, we would have to have moved to the location $x = c(1\text{ s})$. Speed is change in position over change in time. Thus the speed with which we are moving is $x/t = c(1\text{ s})/(1\text{ s}) = c$.

In these notes c will typically be used to represent the speed of light in free space. Using the permittivity and permeability of free space, we obtain $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8$ m/s.