A Numerical Comparison of the Phase Perturbation Technique with the Classical Field Perturbation and Kirchhoff Approximations for Random Rough Surface Scattering

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Abstract—The classical field perturbation and Kirchhoff approximations can be viewed as low and high frequency solutions to the rough surface scattering problem. The former is accurate for surfaces whose roughness is much smaller than the wavelength of the incident field and the latter for surfaces that have large radii of curvature relative to the incident wavelength. However, there is a need for a solution that applies to surfaces rough on scales comparable to the incident wavelength. Ideally this solution must also reduce to the two classical solutions in the appropriate limits. In this paper we examine the recently-developed phase perturbation technique. We show numerically that for a one-dimensional surface with Gaussian roughness spectrum satisfying Dirichlet boundary conditions the phase perturbation reflection coefficient reduces to that of field perturbation theory for small surface roughness and to that of the Kirchhoff approximation for gently undulating surfaces. It is also numerically shown that the phase perturbation backscattering coefficient reduces to those of first-order field perturbation theory and the Kirchhoff approximation in the appropriate limits.

INTRODUCTION

Many engineering and scientific problems arise which require a thorough understanding of wave scattering from rough surfaces. For example, knowledge of rough surface scattering is important in areas such as remote sensing, integrated optics, radio communications, oceanography, radio astronomy, surface profilometry, underwater acoustics, and surface physics [1–4].
The classical field perturbation (FP) [5] and Kirchhoff (KA) [6] approximations can be viewed as low and high frequency solutions to the rough surface scattering problem. The former is accurate for surfaces whose roughness is much smaller than the wavelength of the incident field and the latter for surfaces that have large radii of curvature relative to the incident wavelength. However, there is a need for a solution that applies to surfaces rough on scales comparable to the incident wavelength. Ideally this solution must also reduce to the two classical solutions in the appropriate limits. In the past few decades much effort (for example, see Refs. 7 through 14) has gone into devising a solution to "bridge the gap" between these two solutions. However, it has been difficult to obtain a useful approximation that results in numerically tractable expressions.

Recently, Winebrenner and Ishimaru introduced the phase perturbation (PP) technique [15,16] for scattering from rough surfaces. This technique is based on expansion of a function related to the complex phase of the surface source density (as originally proposed by Shen and Maradudin [17]) and has some similarity to a method proposed by Shin [18]. In Ref. 15 Winebrenner and Ishimaru investigated scalar-wave scattering for a periodic surface with Dirichlet boundary conditions. In Ref. 16 they extended the phase perturbation technique to random rough surfaces with Dirichlet boundary conditions. They showed that the phase perturbation expression for the reflection coefficient reduces to that of field perturbation theory for small surface roughness and to that of the Kirchhoff approximation for gently undulating surfaces. Their analysis also indicated that the backscattering coefficient reduces exactly to that of first-order field perturbation theory and approximately to that of the Kirchhoff approximation for gently undulating surfaces. Numerical results for the coherent reflection coefficient for a two-dimensional (2-D) random rough surface were presented. However, because of numerical difficulties, reliable results for the 2-D backscattering coefficient were not obtained.

In this paper we concentrate on the 1-D surface. We show numerically that for a 1-D surface having a Gaussian roughness spectrum and satisfying Dirichlet boundary conditions, the phase perturbation reflection coefficient reduces to that of field perturbation theory for small surface roughness and to that of the Kirchhoff approximation for gently undulating surfaces. It is also numerically shown that the phase perturbation backscattering coefficient reduces to those of first-order field perturbation theory and the Kirchhoff approximation in the appropriate limits.

In the next section a brief review of the phase perturbation technique is given. This is followed by presentation of numerical results for both the coherent reflection coefficient and the incoherent backscattering coefficient. Energy conservation is also considered.

A REVIEW OF PHASE PERTURBATION THEORY

In this section we will review the phase perturbation technique for a 1-D surface. For a comprehensive treatment of phase perturbation for a 2-D surface the reader
should consult Refs. 15, 16, or 19.

Consider a plane wave incident on a one-dimensional surface described by the function \( z = f(x) \). The incident angle is given by \( \theta_0 \) and the incident wave vector \( k_i \) is parallel to the \( x-z \) plane (Figure 1). The half-space above the surface is Region I and the half-space below the surface Region II. We are interested in the field scattered by the surface in Region I. A Dirichlet boundary condition is assumed such that the field vanishes everywhere on the surface. In addition, a harmonic time dependence of \( \exp(-i\omega t) \) is assumed.

![Figure 1. Scattering geometry.](image)

Following the development of Winebrenner and Ishimaru [16] we start with the Helmholtz integral formulation of the extinction theorem. This leads to the definition of a transition matrix, or T-matrix, which has as an unknown the surface source density. This source density is rewritten in exponential form with the unknown argument expanded in a perturbation series. The coherent reflection coefficient and backscattering coefficient are found by taking averages of the coherent and incoherent T-matrices.

Using the boundary and radiation conditions in the Helmholtz integral we find that for a point in Region I the field is given by

\[
\psi(z) = \exp[ik_iz - i\kappa_izz] - \int_{-\infty}^{\infty} dz' G_0(r, r') n' \cdot \nabla' \psi(r') \quad r \in \text{Region I} \quad (1a)
\]

In Region II

\[
0 = \exp[ik_iz - i\kappa_izz] - \int_{-\infty}^{\infty} dz' G_0(r, r') n' \cdot \nabla' \psi(r') \quad r \in \text{Region II} \quad (1b)
\]

\( G_0 \) is the 2-D free-space Green's function given by \( G_0 = (i/4)N_0^{(1)}(k|z - z'|) \), \( r' = (x', f(x')) \) is a point on the surface, \( r = (x, z) \) is the field point, \( k_i = (k_{iz}, -\kappa_{iz}) \) is the incident wave vector, and \( n' \) is a vector normal to the surface and is given by \( n' = \hat{x} - \hat{z} f(x')/dz' \). Equation (1b) is known as the extinction theorem [20].

The first term on the right-hand side of (1a) is the incident plane wave and the
second term the scattered wave $\psi_s(x)$. We want to solve (1b) for the unknown surface source density $n' \cdot \nabla' \psi(x')$ and make use of the solution in (1a) to find the total field $\psi(x)$ in Region I. The free-space Green's function is written in a Weyl plane-wave expansion [21]. Then considering only observation points above the highest point of the surface the scattered wave component of (1a) becomes

$$\psi_s(x) = \int_{-\infty}^{\infty} dk_z \exp[i k_z z + i k_z z] T(k, k_z)$$

(2)

where $k_z$ and $k_z$ are the $x$- and $z$-components of the scattered wave vector $k$ such that $k \cdot x = k_z z + k_z z$ and $k_z = \sqrt{k^2 - k_z^2}$. The square root is taken to be positive for $k_z > k_z$ and positive imaginary for $k_z < k_z$. The quantity

$$T(k, k_z) = -\frac{i}{4\pi k_z} \int_{-\infty}^{\infty} dx' \exp[-ik_z z' + ik_z f(x')] n' \cdot \nabla' \psi(x')$$

(3)

is called the transition or T-matrix [4]. Equation (2) shows that the scattered field is given by the sum of all plane and evanescent waves weighted by a factor $T(k, k_z)$ which gives the coupling from the original propagating direction $k_i$ into each new propagating direction $k$.

The next step is to write the incident plane wave in terms of a delta function and combine this with the Weyl plane-wave expansion for $G_0$ in (1b). Then considering points below the lowest point of the surface yields

$$\delta(k_z - k_{iz}) = \frac{i}{4\pi k_z} \int_{-\infty}^{\infty} dx' \exp[-ik_z z' + ik_z f(x')] n' \cdot \nabla' \psi(x')$$

(4)

Finally, the unknown surface source density is defined to be the product of the source density for a flat surface at $z = 0$ and a correction factor $y(x')$ which compensates for the surface roughness

$$n' \cdot \nabla' \psi(x') = \exp[i k_{iz} z'] y(x')$$

(5)

Our new unknown representing the surface source density is $y(x')$, and our new goal is to find $y(x')$ and use it to find the scattered field $\psi_s(x)$.

Equations (2)--(5) are exact and are the starting point for using the extinction theorem formulation to develop the field perturbation [22] and phase perturbation approximations. In field perturbation theory the unknown correction factor $y(x')$ is expanded in a perturbation series with small parameter $kh$

$$y(x') = y^{(0)}(x') + (kh)y^{(1)}(x') + \frac{(kh)^2}{2} y^{(2)}(x') + \frac{(kh)^3}{3!} y^{(3)}(x') + \ldots$$

(6)

where the constants are included for convenience. In phase perturbation theory the unknown correction factor $y(x')$ is rewritten in exponential form and the complex phase $p(x')$ is expanded in a perturbation series with small parameter $kh$

$$y(x') = \exp[p(x')]$$

$$= \exp[p^{(0)}(x') + (kh)p^{(1)}(x') + \frac{(kh)^2}{2} p^{(2)}(x') + \ldots]$$

(7)

The phase perturbation method is analogous to the Rytov method for wave propagation through a random medium in which the field $U(x)$ is written as
$U(r) = \exp[\phi(r)]$ [1]. Now by expanding (7) in a Maclaurin series and matching orders in $kh$ with (6) we find the relationship between the terms of the two perturbation series. Thus, for second-order phase perturbation the correction factor $y(z')$ is given by

$$y(z') \equiv \exp \left\{ (kh)y^{(1)}(z') + \frac{(kh)^2}{2} \left[ y^{(2)}(z') - (y^{(1)}(z'))^2 \right] \right\}$$  \hspace{1cm} (8)

and combining (3), (5), and (8) yields the phase perturbation approximation for the T-matrix

$$T(k, k_i) \approx -\frac{1}{2\pi} \frac{\kappa_{iz}}{k_z} \int_{-\infty}^{\infty} dz' \exp[-i(k_z - k_{iz})z'] \exp[-ik_z f(z') + \phi(z')]$$  \hspace{1cm} (9)

**Coherent Reflection Coefficient and Incoherent Scattering Coefficient**

From (2) it is clear that the coherent and incoherent scattered intensities are related to the moments of the T-matrix. Hence, the phase perturbation coherent reflection coefficient is found by taking the first moment of the T-matrix given by (9). The phase perturbation incoherent scattering coefficient is found by taking the first and second moments of the T-matrix given by (9). We express the characteristic functions resulting from the averaging process in terms of cumulants [23] and after some straightforward but lengthy manipulations [16,17] arrive at the reflection and scattering coefficients. The phase perturbation expression for the coherent reflection coefficient is given by

$$< R > = -\exp \left[ -2k\kappa_{iz} \int_{-\infty}^{\infty} dK \frac{W(K) \beta((K + k_{iz})/k)} {K/k} \right]$$  \hspace{1cm} (10)

where $\beta(K/k) = [1 - (K/k)^2]^{1/2}$ and $W(K)$ is the surface spectral density defined by

$$W(K) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{-iKz} B_f(z)$$  \hspace{1cm} (11)

where $B_f(z = z_1 - z_2)$ is the surface correlation function $< f(z_1)f(z_2) >$. It should be noted that this definition for $W(K)$ is slightly different from the one used by Winebrenner and Ishimaru [15,16]. It is normalized to yield $\kappa^2$ when $W(K)$ is integrated over all $K$.

The phase perturbation approximation for the bistatic scattering coefficient is given by

$$\sigma(k, k_i) = \frac{\kappa_{iz}^2}{k} e^{-2Re[N_2]} \int_{-\infty}^{\infty} dz \exp[-i(k_z - k_{iz})z] \left[ e^{N_1(z)} - 1 \right]$$  \hspace{1cm} (12)

where

$$N_2 = \frac{\kappa_{iz}^2}{2} (k_z - k_{iz}) + k(k_z + k_{iz}) \int_{-\infty}^{\infty} dK W(K) \beta((K + k_{iz})/k)$$  \hspace{1cm} (13)
\[ N_{11}(z) = \int_{-\infty}^{\infty} dK W(K) e^{ikz} |k_z + k\beta((K + k_iz)/k)|^2 \] (14)

\( W(K) \) is the surface spectral density as defined above by (11), and \( \beta(K/k) = [1 - (K/k)^2]^{1/2} \).

**NUMERICAL RESULTS**

Winebrenner and Ishimaru [16] showed analytically that the phase perturbation expressions for the reflection and backscattering coefficients reduce to those of field perturbation in the limit as \( kh \), the surface roughness parameter, goes to zero. Furthermore, they showed that the phase perturbation reflection coefficient reduces to the Kirchhoff expression for moderate angles of incidence when the surface roughness is gently undulating on the length scale of the radiation wavelength. In fact, this is when the Kirchhoff approximation is expected to be accurate. More precisely the phase perturbation reflection coefficient reduces to the Kirchhoff expression when, for a given incident angle, the width of the surface roughness spectrum is sufficiently narrow. Analysis by Winebrenner and Ishimaru also indicated that the phase perturbation backscattering coefficient differs from the Kirchhoff expression by a factor of \( \cos^4 \theta_0 \) for gently undulating surfaces. The reduction is exact at normal incidence and is close (to within 1 dB) for incident angles less than or equal to 20°.

In this section we present numerical results for the reflection coefficient and backscattering coefficient and compare them with those of the field perturbation and Kirchhoff approximations. The surface roughness parameter \( kh \) is varied between 0.1 and 2, where \( k \) is the radiation wavenumber and \( h \) the rms surface height, and \( h/l \) is varied between 0.05 and 0.5, where \( l \) is the surface correlation length and \( h/l \) is proportional to the rms slope (= \( \sqrt{2h/l} \)).

Inspection of (12) and (14) shows that calculation of the backscattering coefficient involves a double integral; both integrands have infinite limits of integration and both integrands vary between being nonoscillatory (when \( z = 0 \) or \( K = 0 \)) and being highly oscillatory. For a Gaussian spectral density the integrand of (14) depends on the correlation length \( l \); for large \( l \) the envelope of the integrand is highly peaked and for small \( l \) it tapers gradually. Note that (14) is a function of \( z \); for a Gaussian roughness spectrum its integrand is nonanalytic in \( K \), thus eliminating the use of contour deformation.

Several quadrature methods were considered (Ref. 24 and references therein) and piecewise Gauss quadrature (using Legendre polynomials to generate weights and abscissas) was chosen. Gauss quadrature of order \( n \) gives an exact solution for the integral of a polynomial of order less than or equal to \( 2n - 1 \). While it is not necessarily the most efficient means of evaluating the phase perturbation equations, it is straightforward and easily implemented. For the Gaussian spectral density used, numerical convergence occurred more rapidly for highly peaked envelopes when the interval was divided into many subintervals and a low order of quadrature was chosen; it occurred more rapidly for gradually tapering envelopes
when the interval was divided into a few subintervals and a high order of quadrature was chosen. Quadrature orders up to 256 were used. All calculations were performed on a VAX 11/730 using double precision arithmetic.

Reflection Coefficient

The coherent reflection coefficients for the phase perturbation, field perturbation, and Kirchhoff approximations are given by [22,1]

\[
PP: \quad \langle R \rangle = -\exp[-2k^2 \cos \theta_0 \int_{-\infty}^{\infty} dKW(K)\beta((K+k_{iz})/k)] \tag{15}
\]

\[
FP: \quad \langle R \rangle = -1 + 2k^2 \cos \theta_0 \int_{-\infty}^{\infty} dKW(K)\beta((K+k_{iz})/k) \tag{16}
\]

\[
KA: \quad \langle R \rangle = -\exp[-2(kh \cos \theta_0)^2] \tag{17}
\]

where \( \beta(K/k) = [1-(K/k)^2]^{1/2} \) and \( W(K) \) is the surface spectral density defined by (11). The quantity \( 20 \times \log_{10} \) of the magnitude of the reflection coefficient is plotted for each of the equations above in Figures 2-4 for the Gaussian spectral density

\[
W(K) = \frac{h^2}{2\sqrt{\pi}} e^{-K^2/4} \tag{18}
\]

where \( h \) is the rms surface height and \( l \) is the correlation length. The field perturbation expression is derived from the extinction theorem formulation as given by Nieto-Vesperinas and Garcia [21]. However, the extinction theorem formulation and the Rayleigh-Rice formulation [1,6] have been shown to lead to equivalent results through fifth order [25]. Hence, (16) is equivalent to the classical field perturbation result [6].

Figure 2 illustrates an instance when phase perturbation reduces numerically to both the field perturbation and Kirchhoff approximations in the appropriate limits. In Fig. 2 the reflection coefficient has been calculated as a function of surface roughness ranging from slightly rough to very rough. The angle of incidence is 30°, and the rms slope is 0.05\( \sqrt{2} \) (slope angle = 4.04°).

We expect agreement between phase perturbation and field perturbation for \( kh \ll 1 \) and, in fact, there is agreement for \( kh \lesssim 0.4 \). Inspection of (16) together with (18) shows that as the surface roughness parameter \( kh \) is increased, field perturbation eventually violates energy conservation. This is demonstrated in Fig. 2. In contrast, for phase perturbation there is no obvious violation of energy conservation in Fig. 2.

For a deterministic surface the radius of curvature is given by

\[
\rho = \left[ 1 + \left( \frac{df}{dz} \right)^2 \right]^{3/2} \frac{dz^4}{dz^2} \tag{19}
\]
Figure 2. The magnitude of the reflection coefficient as a function of the surface roughness parameter $kh$ for phase perturbation (PP) (dashed line), field perturbation (FP), and the Kirchhoff approximation (KA). The angle of incidence is 30°, and the slope is 0.05√2 (slope angle = 4.04°).

For a random rough surface with a Gaussian spectrum we define the average radius of curvature to be

$$<\rho> = \left[1 + \left(\frac{\sqrt{2}h}{i}\right)^2\right]^{3/2} \left(\frac{2\sqrt{3}h}{i^2}\right)$$

(20)

The Kirchhoff approximation is generally considered to be valid for moderate angles of incidence when the average radius of curvature is large compared to the incident wavelength — that is,

$$\frac{<\rho>}{\lambda} \gg 1$$

(21)

This condition is reasonably satisfied everywhere in Fig. 2, ranging between 1.85 for $kh = 0.1$ and 27.03 for $kh = 2$. Thus, we expect the Kirchhoff results — and by extension the phase perturbation results — to be correct.

In Fig. 3 the rms slope has been increased tenfold to 0.5√2 (slope angle = 35.26°). Since the Kirchhoff result for the coherent reflection coefficient (17) does not depend on the slope, the Kirchhoff curves in Figs. 2 and 3 are identical. In contrast phase perturbation does depend on the slope and the results are appreciably different. Phase perturbation predicts a higher coherent return than the
Kirchhoff approximation. In fact, this higher coherent return is in qualitative agreement with work done by Thorsos [26] using an exact numerical method to examine the Kirchhoff approximation. Also, the average radius of curvature given by (20) ranges from 0.03λ to 0.68λ so that condition (21) is not satisfied.

In order to increase the surface slope while maintaining the same surface heights we have decreased the correlation lengths in Fig. 3. Thus, there is more energy in the higher spatial frequencies of the surface spectrum — that is, increasing the surface slope increases the width of the surface spectrum — and phase perturbation no longer reduces to the Kirchhoff approximation.

![Graph](image)

**Figure 3.** The magnitude of the reflection coefficient as a function of the surface roughness parameter $kh$ for phase perturbation (PP) (dashed line), field perturbation (FP), and the Kirchhoff approximation (KA). The angle of incidence is the same as in Figure 2 — 30° — but the slope has been increased tenfold to $0.5\sqrt{2}$ (slope angle = 35.26°).
Figure 4. The magnitude of the reflection coefficient as a function of the surface roughness parameter \(kh\) for phase perturbation (PP) (dashed line), field perturbation (FP), and the Kirchhoff approximation (KA). The angle of incidence has been increased to 85\(^\circ\), and the slope is the same as in Figure 2 — 0.05\(\sqrt{2}\) (slope angle = 4.04\(^\circ\)).

In Fig. 4 we again consider a surface with an rms slope of 0.05\(\sqrt{2}\) (slope angle = 4.04\(^\circ\)) but this time use an incident angle of 85\(^\circ\) which is equal to a low grazing incident angle of 5\(^\circ\). At low grazing angles the Kirchhoff approximation can no longer be expected to give accurate results. Also as indicated in Ref. 16 an increase in the angle of incidence requires an even narrower surface spectrum in order for phase perturbation to reduce to the Kirchhoff approximation. In Fig. 4 the spectrum is insufficiently narrow so that the results of the two methods disagree.

In Fig. 4 the Rayleigh roughness parameter \(2kh \cos \theta_0\) has a maximum value of 0.35 when \(kh = 2\). The large incident angle decreases the amount of phase difference between two different scattering points so that the surface appears less rough and the incoherent scatter is reduced. Phase and field perturbation agree well everywhere. However, when the slope is increased tenfold (Figure 5), the two no longer agree as well.
Figure 5. The magnitude of the reflection coefficient as a function of the surface roughness parameter $k h$ for phase perturbation (PP) (dashed line), field perturbation (FP), and the Kirchhoff approximation (KA). The angle of incidence is the same as in Figure 4 — $85^\circ$ — but the slope has been increased tenfold to $0.5\sqrt{2}$ (slope angle $= 35.26^\circ$).

Backscattering Coefficient

In Figs. 6-8 we have plotted the backscattering coefficients [22,1]

\[ PP: \quad \sigma(\theta_0) = k \cos^2 \theta_0 e^{-2Re[N_2]} \int_{-\infty}^{\infty} dz \exp[i2k_{iz}z][e^{N_{11}(z)} - 1] \]  

(22)

\[ N_2 = 2k^2 \cos \theta_0 \int_{-\infty}^{\infty} dKW(K)\beta((K + k_{iz})/k) \]  

(22a)

\[ N_{11}(z) = \int_{-\infty}^{\infty} dKW(K)e^{iKz}[k \cos \theta_0 + k\beta((K + k_{iz})/k)]^2 \]  

(22b)

\[ FP: \quad \sigma(\theta_0) = 8\pi k^3 \cos^4 \theta_0 W(2k_{iz}) \]  

(23)

\[ KA: \quad \sigma(\theta_0) = \frac{k}{\cos^2 \theta_0} e^{-(2kh \cos \theta_0)^2} \int_{-\infty}^{\infty} dz \exp[i2k_{iz}z][e^{(2k \cos \theta_0)^2B_f(z)} - 1] \]

\[ = \frac{\sqrt{\pi}k^l}{\cos^2 \theta_0} e^{-(2kh \cos \theta_0)^2} \sum_{m=1}^{\infty} \frac{(2kh \cos \theta_0)^{2m}}{m!\sqrt{m}} \exp[-k_{iz}^2/2/m] \]  

(24)

in decibels for the Gaussian spectral density $W(K)$ given by (18). Again we can
consider (23), derived from the extinction theorem formulation, to be the classical perturbation result [27].

In Fig. 6 the backscattering coefficient has been calculated as a function of the incident angle for a slightly rough surface \([kh = 0.1\) and slope angle = 4.04°\]. Phase perturbation and field perturbation agree everywhere, consistent with our expectations. On the other hand the phase perturbation and Kirchhoff results differ except at small angles of incidence. In Fig. 6 the average radius of curvature given by (20) is 1.85λ, and if we assume the field perturbation results to be correct, (21) is not satisfied.

In Fig. 7 the roughness has been increased ten times \([kh = 1]\) and the slope three times \([= 0.15\sqrt{2}]\) so that field perturbation is no longer valid. The average radius of curvature is now 2.18λ. While this value is not much greater than that of Fig. 6, recent work by Thorsos [28] supports the validity of the Kirchhoff results shown in Fig. 7. Using an exact numerical method Thorsos found that the valid region for the Kirchhoff approximation depends more directly on the correlation length than on the radius of curvature. In Fig. 6 \(kl = 2\) \((k = \text{wavenumber}, \ l = \text{correlation length})\) and in Fig. 7 \(kl = 6.67\). Figures 6 and 7 show that agreement between phase perturbation and the Kirchhoff approximation depends on the correlation length.

![Figure 6. The backscattering coefficient as a function of incident angle \(\theta_0\) for phase perturbation (PP) (dashed line), field perturbation (FP), and the Kirchhoff approximation (KA). The surface roughness parameter \(kh\) is 0.1, and the slope is 0.05\(\sqrt{2}\) (slope angle = 4.04°).](image-url)
Earlier analysis [16] indicated that for a sufficiently narrow surface spectrum the phase perturbation backscattering coefficient differs from the Kirchhoff expression by a factor of \( \cos^4 \theta_0 \). However, in Fig. 7 the phase perturbation and Kirchhoff results agree very closely over all angles. This apparent discrepancy between analytical and numerical results will be discussed in a later section. In practice we have found that the phase perturbation results reduce to the Kirchhoff results over a large range of parameter values when the Kirchhoff approximation is considered to be valid.

The last figure (Fig. 8) illustrates a case when the Kirchhoff approximation and phase perturbation give entirely different results. The surface roughness parameter is the same as that of Fig. 7 \([kh = 1]\) but the slope has been doubled so that the surface is effectively rougher. The average radius of curvature is now 0.65\( \lambda \) and (21) is not satisfied. Work done [26,29] using exact numerical methods indicates that at larger angles of incidence the lower incoherent return given by phase perturbation is qualitatively correct. Figure 8 again demonstrates that agreement between phase perturbation and the Kirchhoff approximation depends on the correlation length. In Fig. 8 the correlation length is half that of the correlation length in Fig. 7.

![Figure 7](image.png)

**Figure 7.** The backscattering coefficient as a function of incident angle \( \theta_0 \) for phase perturbation (PP) (dashed line), field perturbation (FP), and the Kirchhoff approximation (KA). The surface roughness parameter has been increased tenfold to \( kh = 1 \) and the slope to \( 0.15\sqrt{2} \) (slope angle = 11.98°).
Figure 8. The backscattering coefficient as a function of incident angle $\theta_0$ for phase perturbation (PP) (dashed line), field perturbation (FP), and the Kirchhoff approximation (KA). The surface roughness parameter is the same as in Figure 7 — $kh = 1$ — but the slope has been doubled to $0.3\sqrt{2}$ (slope angle = 22.99°).

Energy Conservation

An attempt was made to investigate energy conservation for two different scales of surface roughness — $kh = 0.1$ and $kh = 1$. We were able to show energy conservation for a surface with $kh = 0.1$ and $h/l = 0.2$ to within 0.002% for an incident angle of 89.5° and to within 3% for an incident angle of 30°. In the case of small roughness most of the scattered energy is coherent. For $kh = 1$ energy conservation could not be shown because of difficulty with numerical convergence at low scattered grazing angles.

Analytical vs. Numerical Results

For a gently undulating surface with a sufficiently narrow surface roughness spectrum and for small angles of incidence, we can expand the $\beta$ terms appearing in (22a) and (22b) in a Maclaurin series

$$\beta((K + k_{iz})/k) = \cos \theta_0 - \tan \theta_0 \frac{K}{k} - \frac{1}{2} \frac{1}{\cos^3 \theta_0} \frac{K^2}{k^2} + \cdots$$  \hspace{1cm} (25)
If we retain only the first term, the phase perturbation backscattering coefficient (22) reduces [16] to the Kirchhoff backscattering coefficient (24) times a factor of \( \cos^4 \theta_0 \). We will refer to this reduction as the phase-Kirchhoff result. In order to understand the apparent discrepancy between this result and our numerical results we must examine what happens when we retain the second term in (25), as well as the first. Equations (22a) and (22b) become

\[
N_2 \equiv 2(kh)^2 \cos^2 \theta_0 \tag{26a}
\]

\[
N_{11}(x) \equiv 4(kh)^2 \cos^2 \theta_0 e^{-x^2/l^2} - i \frac{8(kh)^2 \sin \theta_0 x}{kl} \frac{e^{-x^2/l^2}}{l} \tag{26b}
\]

\[
\equiv \text{Re}(N_{11}) + i \text{Im}(N_{11})
\]

for the Gaussian surface roughness spectrum given by (18). \( N_2 \) and \( \text{Re}(N_{11}) \) are due to the first term in (25) and \( \text{Im}(N_{11}) \) is due to the second. Noting that \( \text{Re}(N_{11}) \) is an even function of \( x \) and \( \text{Im}(N_{11}) \) is an odd function of \( x \), we can write the backscattering coefficient (22) approximately as

\[
\sigma(\theta_0) \approx k \cos^2 \theta_0 \int_{-\infty}^{\infty} dx \left\{ \cos(2kz \sin \theta_0 + \text{Im} N_{11}) e^{\text{Re} N_{11} - 2 \text{Re} N_2} \right. \\
- \left. \cos(2kz \sin \theta_0) e^{-2 \text{Re} N_2} \right\} \\
\approx k \cos^2 \theta_0 \int_{-\infty}^{\infty} dx \left\{ \cos[2kz \sin \theta_0 (1 - 4(h/l)^2 e^{-x^2/l^2})] e^{-x^2(1-\exp(-x^2/l^2))} \\
- \cos(2kz \sin \theta_0) e^{-x^2/l^2} \right\} \tag{27}
\]

where

\[ \chi = 2kh \cos \theta_0 \]

When the quantity \( 4(h/l)^2 e^{-x^2/l^2} \) (due to the second term in (25)) can be neglected, (27) reduces to the phase-Kirchhoff result. However, for constant \( h/l \) (27) never reduces to the phase-Kirchhoff result even when \( kh \) and \( kl \) are taken to infinity (the geometric optics limit).

It would seem upon inspection of (27) that for a fixed surface height \( h \) we can increase the correlation length \( l \) until eventually the quantity \( 4(h/l)^2 e^{-x^2/l^2} \) becomes negligible. However, extensive numerical investigation does not support this. For example, for \( kh = 2 \), \( kl = 100 \), and \( \theta_0 = 15^\circ \) (27) has still not reduced numerically to the phase-Kirchhoff result. In fact, it has reduced to the Kirchhoff approximation. Apparently this occurs because as we increase the correlation length \( l \) for a fixed height \( h \), the surface becomes smoother and backscattering decreases. As a consequence, the contribution from \( 4(h/l)^2 e^{-x^2/l^2} \) remains relatively significant even as \( 4(h/l)^2 e^{-x^2/l^2} \) decreases in magnitude.

From the above analysis we conclude that higher order terms in (25) cannot be neglected even for a relatively narrow surface roughness spectrum. In the examples studied numerically it is precisely these terms which compensate for the \( \cos^4 \theta_0 \) factor and cause phase perturbation to reduce to the Kirchhoff approximation.
CONCLUSIONS

We have shown that the phase perturbation expressions for the reflection coefficient and backscattering coefficient are numerically tractable for the case of a one-dimensional surface with Gaussian roughness spectrum satisfying Dirichlet boundary conditions. Furthermore, the phase perturbation results reduce numerically to those of both field perturbation theory and the Kirchhoff approximation when these methods are considered to be accurate. Agreement with the two limiting cases strongly suggests that phase perturbation will be useful for treating surfaces which do not satisfy the constraints of the two classical methods. In fact, the phase perturbation results differed in several cases when neither field perturbation nor Kirchhoff theory is expected to give good results. Of course, the accuracy of the phase perturbation results still must be determined.

We were also able to demonstrate numerically that phase perturbation obeys energy conservation for surfaces with small roughness. However, for surfaces with large roughness energy conservation was not shown because of difficulty with numerical convergence.

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