A Comparison of Perturbation Theory and the Small-Slope Approximation for Acoustic Scattering From a Rough Interface for a Biot Medium

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Abstract—In this paper, the lowest order small-slope approximation (SSA) scattering cross section for Biot theory is derived. Numerical results are obtained for both backscattering and bistatic scattering using a modified power law spectrum, and these results are compared with those of lowest order perturbation theory (PT). Frequencies ranging from 100 Hz to 3 kHz are used for surfaces with rms heights $h$ of 0.1 and 1 m and a correlation length $l$ of 10 m. The angle of incidence for the bistatic results is limited to $45^\circ$. It is found that for the smaller surface height roughness ($h = 0.1$ m), the SSA and PT give the same results for frequencies up to almost 1 kHz for both backscattering and bistatic scattering. For $h = 1$ m, the SSA and PT backscatter results are in good agreement at all frequencies for incident grazing angles up to approximately $45^\circ$. For the bistatic results, the SSA and PT results agree only at low grazing angles of scatter. In the specular region, the results differ significantly.

Index Terms—Biot theory, fluid-fluid-saturated porous solid, perturbation theory, rough surface scattering, small-slope approximation.

I. INTRODUCTION

In an effort to provide a more realistic and accurate model for acoustic scattering from the ocean bottom, researchers have turned to Biot theory to model the sediment. According to Biot theory, a sediment environment such as the ocean bottom can be described as a fluid-saturated, porous solid [1], [2]. Two coupled equations govern the motion of the fluid and of the porous solid skeletal frame. In addition to the compressional and shear waves occurring in an elastic solid, Biot theory predicts a third, or slow wave, in the sediment. The Biot model has been the subject of a number of studies [1]–[9], several of which concern scattering from an interface between a fluid and a Biot medium. Stoll and Kan [5] presented a comprehensive study on wave reflection in which they treated the ocean bottom as a flat interface between a fluid and Biot medium. Wu et al. [7] extended the work of Stoll and Kan for the case of general boundary conditions; they also considered a wave incident from

the Biot medium. Williams et al. [9] derived perturbation results based on Biot theory for scattering from randomly rough surfaces and obtained numerical results for a power law spectrum.

In the study of rough surface scattering, much research has been directed toward analytic models for backscattering and bistatic scattering that are applicable beyond the two classical approaches, the perturbation method and the Kirchhoff approximation. The small-slope approximation (SSA), proposed by Voronovich [10], gives a systematic series expansion in generalized surface slope. The SSA expansion of the transition, or $T^*$- matrix satisfies the reciprocity condition at each order and reduces to the standard perturbation series for small surface roughness. When the SSA $T^*$-matrix is found to second order in generalized slope, it reduces to that of the Kirchhoff approximation in the high frequency limit. Several numerical studies of the SSA for acoustic scattering from rough surfaces have appeared in the literature. Berman compared Monte Carlo SSA and exact results for bistatic scattering for both pressure-release and fluid-solid boundaries for one-dimensional (1-D) surfaces satisfying a Pierson Moskowitz spectrum or a modified power spectrum [11]. Broschat compared formally averaged (theoretical) results for the reflection coefficient with exact results for a Pierson Moskowitz power law spectrum [12]. Yang and Broschat presented bistatic scattering results for 2-D, pressure-release surfaces for a Gaussian spectrum [13] as well as bistatic scattering results for 1-D fluid–elastic–solid surfaces for both Gaussian and modified power law spectra [14]. The SSA theory was examined in detail by Thoros and Broschat for pressure-release surfaces [15]. In their companion paper, Broschat and Thoros examined the accuracy of the SSA for 1-D, pressure-release surfaces by comparison with exact numerical results for surfaces with Gaussian statistics and a Gaussian roughness spectrum [16]. It was found that, for surfaces with rms slope angles up to $45^\circ$, the SSA results agree well with the exact results over a broad range of angles of scatter.

This paper extends the SSA to acoustic scattering from a rough interface between a fluid and a fluid-saturated, porous solid (Biot medium). The equation for the lowest order SSA scattering cross section is derived, and numerical results are presented for 1-D surfaces satisfying a modified power law spectrum. Results for both the backscattering cross section and the bistatic scattering cross section are compared with results for lowest order perturbation theory (PT) for frequencies ranging from 100 Hz to 3 kHz. Fixed rms surface heights $h$ of 0.1 and 1 m and a correlation length $l$ of 10 m are used. In terms of the dimensionless parameters, $k_{\varphi}h$ and $k_{\varphi}l$, the frequency range and
surface heights used correspond to \(0.04 \leq k_w h \leq 12.57\) and \(4.19 \leq k_w l \leq 125.66\) where \(k_w\) is the incident wavenumber. It is found that for small surface height roughness (\(h = 0.1\) m), the SSA and PT give the same results for frequencies up to almost 1 kHz for both backscattering and bistatic scattering (\(0.04 \leq k_w h \leq 0.42\)). At a frequency of 1 kHz, the bistatic scatter results start differing only near the specular region; this corresponds to incidence angles near normal for backscatter. For \(h = 1\) m, the SSA and PT backscatter results are in good agreement at all frequencies for incident grazing angles up to approximately 45°. However, as the frequency is increased, the differences become substantial for larger grazing angles. For the bistatic results, the SSA and PT results agree only at low grazing angles of scatter for the lower frequencies. As the frequency is increased, there is still agreement at low grazing angles in the back direction, but over a broad range of angles, the differences become large; at 3 kHz, for example, the difference is greater than 40 dB in the specular direction.

In the next section, the expression for the lowest order scattering cross section based on the SSA is derived. In Section III, numerical results for the backscattering and bistatic scattering strength are given for the SSA and for lowest order perturbation theory. The results are summarized in Section IV.

II. DERIVATION OF THE SSA SCATTERING CROSS SECTION

The study of acoustic scattering from a rough interface between a fluid and a fluid-saturated, porous solid using Biot theory requires a description of the parameters in the Biot medium occupying the lower half space beneath the rough surface. For the scattering geometry shown in Fig. 1, it is assumed that the lower medium is composed of two components, a poro-elastic lattice (solid frame) and a saturating fluid (interstitial fluid), and that this lower medium is macroscopically homogeneous and isotropic. It is also assumed that the volume in consideration is large relative to the size of the grains and pores of the solid frame and that the fluid can possess nonzero viscosity and can move relative to the frame. Two coupled wave equations describe the displacement of the two components.

A plane wave is assumed to be incident on the interface from the upper medium which is the same fluid as the interstitial fluid in the lower medium. Part of the incident wave energy is transmitted into the lower half space, and part is scattered back into the upper half space. The energy transmitted into the lower half space is distributed into three different waves—two compressional waves, “fast” and “slow,” and a shear wave. The slow wave is a consequence of the relative motion and coupling between the saturating fluid and the solid frame; the shear wave can have two components perpendicular to the direction of wave propagation, the “vertically” polarized shear wave and a “horizontally” polarized shear wave excited by the surface roughness. The displacement fields of interest can thus be written as

\[
\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_e + \mathbf{u}_h
\]

(1)

\[
\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2 + \mathbf{U}_v + \mathbf{U}_h
\]

(2)

in the lower medium and

\[
\mathbf{U}_w = \mathbf{U}_v + \mathbf{U}_s
\]

(3)

in the upper medium, where the 1, 2, \(v, h, i,\) and \(s\) subscripts indicate the fast and slow compressional waves, the two shear wave components, the incident wave, and the scattered wave, respectively, and where \(\mathbf{u}\) is the displacement field of the solid frame, \(\mathbf{U}\) is the displacement field of the interstitial fluid, and \(\mathbf{U}_w\) is the fluid displacement field in the upper half space. The relationship between the known incident field and all other displacement fields is dictated by the boundary conditions: continuity of fluid motion normal to the surface, equilibrium of tractions, and equilibrium of fluid pressure [5].

In this paper, the quantity of interest is the scattering cross section. The scattering cross section quantifies the energy scattered from the rough interface between the upper fluid half space and the Biot medium and is therefore related to the scattered displacement field \(\mathbf{U}_s\). It can be written in terms of the \(T\)-matrix [17]

\[
\sigma \delta (\mathbf{K}_i - \mathbf{K}_s') = k_z^2 \left( \langle T(\mathbf{K}_s,\mathbf{K}_i) T^*(\mathbf{K}_s,\mathbf{K}_s') \rangle - \langle T(\mathbf{K}_s,\mathbf{K}_i) \rangle \langle T^*(\mathbf{K}_s,\mathbf{K}_s') \rangle \right)
\]

(4)

where \(\mathbf{K} = (k_x, k_y, k_z)\), \(k_z = \sqrt{k^2 - K^2}\), \(K = |\mathbf{K}|\), the subscripts \(i\) and \(s\) indicate the angles of incidence and scatter, respectively, the angle brackets indicate ensemble averages, an asterisk indicates the complex conjugate, and the right-hand side of (4) is proportional to \(\delta (\mathbf{K}_i - \mathbf{K}_s')\).

The \(T\)-matrix can be expanded in a perturbation series in powers of \(k_w h\) where \(k_w\) is the incident wavenumber. Williams et al. used PT to obtain the first two series terms for the scattering amplitude \(A\) which is directly related to the \(T\)-matrix by [9]

\[
A(\mathbf{K}_s,\mathbf{K}_i) = k_z^2 T(\mathbf{K}_s,\mathbf{K}_i).
\]

(5)

To facilitate comparison between this paper, the work of Williams et al. [9], and previous papers on the SSA by Yang, Broschat, or Thorsos [12]–[16], the notation used in this paper follows that of Williams et al. with the exception of the use of the \(T\)-matrix and the use of \(f(R)\) and \(F(K)\) for the surface profile and its Fourier transform, respectively.
where the final subscript 1 on the right-hand sides of these two equations indicates the first term of a $5 \times 1$ column vector. This term represents scattering into the upper half space, while the remaining four terms of the vector represent scattering into the Biot medium. The terms $G_{0,1}$ and $G_{1,1}$ have been introduced to simplify the notation for the derivation below. In (6) and (7), $Y_1$ is a $5 \times 5$ matrix, $S_0$ is a $5 \times 1$ column vector which is comprised of the reflection coefficient and transmission coefficients for a flat interface between a fluid and Biot medium, $F^{(3)}$ is a $5 \times 5$ matrix, $B$ is a $5 \times 6$ matrix, and $D_0$ is a $6 \times 1$ column vector. The final term $F(K)$ in (7) is the Fourier transform of the surface profile. The expressions for the matrices and vectors are given in the Appendix.

As shown previously (see, for example, [14], [16]), the first two terms of the SSA $T$-matrix can be determined using (6) and (7). In the SSA the $T$-matrix assumes the form

$$T_{0,1}^{SSA}(K_0, K_i) = \frac{-i}{(2\pi)^{2}v_z} \int dR e^{ik_0zR} \frac{1}{\Phi(K_0, K_i)}$$

(8)

where $v_z = \kappa_{z} + k_{z2}$, $k_{z2} = k_{w}\cos\theta_w$, $k_{z} = k_{w}\cos\theta_w$, $\theta_w$ is the angle of incidence in the $x_2z$ plane, $\theta_w$ is the zenith angle of scatter, $R = (x, y)$, and the limits of integration here and for all subsequent integrals are $\pm\infty$. In (8), the phase factor has been explicitly extracted to account for both horizontal and vertical translations of the interface. The remaining term $\Phi(K_0, K_i)$ is a functional Taylor series in $F(K)$ and is given by

$$\Phi(K_0, K_i) = \Phi_0(K_0, K_i) + \int dK_1 e^{ik_1zR} F(K_1) \cdot \Phi_1(K_0, K_i, K_1) + \int dK_1 dK_2 e^{ik_1zR} F(K_1) F(K_2) \Phi_2(K_0, K_i, K_1, K_2) + \cdots.$$  

(9)

To determine the coefficients $\Phi_n$, the exponential term in (8) $e^{-i\kappa_0x f(R)}$ is expanded in powers of the surface height and compared order by order with the $T$-matrix series for PT [15]. We are interested in finding the lowest order term, $\Phi_0$. We write

$$T_{0,1}^{SSA}(K_0, K_i) \approx \frac{-i}{(2\pi)^{2}v_z} \int dR e^{ik_0zR} e^{-i\kappa_0x f(R)} \Phi_0$$

(10)

where the second approximation in (10) is obtained by retaining the first two terms of the expansion for $e^{-i\kappa_0x f(R)}$. Comparison of the zeroth-order term of (10) with (6) gives

$$\Phi_0 = -G_{0,1}(K_0)$$

(11)

but only for $K_0 = K_i$. $\Phi_0$ must still be specified for $K_0 \neq K_i$. This is accomplished by comparing the first-order term of (10) with (7), yielding

$$\Phi_0 = -i \frac{v_z}{2\pi} G_{1,1}(K_0, K_i)$$

(12)

The equations (11) and (12) are consistent, with (12) reducing to (11) for $K_0 = K_i$. The lowest order term of the SSA $T$-matrix is then

$$T_{0,1}^{SSA}(K_0, K_i) = \frac{i}{(2\pi)^{2}v_z} G_{1,1}(K_0, K_i) \int dR e^{i(k_0z-R)e^{-i\kappa_0x f(R)}},$$

(13)

In obtaining (10), a first-order term associated with $\Phi_1$ has been neglected. However, as discussed in [15] for the Dirichlet problem, this term can be transformed into a zeroth-order term which does not contribute to $\Phi_0$.

Thus far, results have been presented for 2-D surfaces. Since the numerical results presented in the next section are for 1-D surfaces, the remaining equations are given for 1-D surfaces with $K_0 = k_{sc} = k_{w}\sin\theta_s$ and $K_i = k_{ic} = k_{w}\sin\theta_i$. Equation (13) becomes

$$T_{0,1}^{SSA}(k_{sc}, k_{ic}) = \frac{i}{2\pi v_z} G_{1,1}(k_{sc}, k_{ic}) \int dx e^{i(k_{sc}x-R)e^{-i\kappa_0x f(x)}}.$$  

(14)

The scattering cross section for 1-D surfaces is given by [17]

$$\sigma(\kappa_{ic} - \kappa_{ic}) = \frac{k_{ic}^2}{k_{w}} \frac{2}{(2\pi)^2} \int \int [T(k_{sc}, k_{ic})T^*(k_{sc}, k_{ic})]$$

(15)

where $T_{0,1}^{SSA}(k_{sc}, k_{ic}) = \langle T_0T^*_0 \rangle - \langle T_0 \rangle \langle T_0^* \rangle$.

In (17), the moments on the right-hand side yield a delta function that cancels the delta function on the left-hand side. Taking the first and second moments of (14) leads to

$$\sigma^{SSA}(k_{sc}, k_{ic}) = \frac{k_{ic}^2}{2\pi k_{w} v_z} \left| G_{1,1}(k_{sc}, k_{ic}) \right|^2$$

(18)

where $v_z = k_{ic} - k_{sc}$ and $C(x)$ is the surface correlation function $\langle f(x_1)f(x_1+X) \rangle / h^2$. At first inspection, (18) is not man-
ifestly reciprocal. However, a $\kappa_{32}$ term can be extracted from the $G_{1,1}$ term. Also, because of the complexity of the matrices given in the Appendix, numerical tests were performed with two different incident angles to insure reciprocity was achieved numerically.

III. NUMERICAL RESULTS

In this section, numerical results are presented for the lowest order SSA and PT backscattering and bistatic scattering strengths using a modified power law spectrum. The scattering strength is related to the scattering cross section by

$$SS = 10\log \sigma(k_{ax}, k_{i2}) \text{[dB]}$$

and the backscattering cross section is obtained from the bistatic scattering cross section by replacing all $k_{ax}$ by $-k_{i2}$. The expression for the lowest order PT scattering cross section is obtained using (7) for 1-D surfaces in (15) and taking the first and second moments of $T_1^{PT}(k_{ax}, k_{i2})$. This gives

$$\sigma^{PT}(k_{ax}, k_{i2}) = \frac{k_{i2}^2}{k_w} |G_{1,1}(k_{ax}, k_{i2})|^2 W(k_{ax} - k_{i2})$$

where $W(K)$ is the surface roughness spectrum defined for 1-D surfaces by

$$W(K) = \frac{1}{2\pi} \int dx e^{-ikx} (f(x_0)f(x + x_0)).$$

The SSA scattering cross section reduces to (20) in the limit as $\kappa_i h \rightarrow 0$. The roughness spectrum used in this work is the modified power law spectrum given by

$$W(K) = \frac{A}{(K^2T^2 + 1)^2}$$

where $A = 2l^2l/\pi$ is the spectral strength. The parameter $l$ controls the fall-off of the correlation function and is referred to as the correlation length in this work. Equation (22) is consistent with the spectrum used in [18]. The corresponding correlation function is given by

$$C(x) = \sqrt{\frac{2}{\pi}} \left(\frac{x}{l}\right)^{3/2} B_{K,3/2} \left(\frac{x}{l}\right)$$

where the nonstandard $B_{K,3/2}$ is used to denote the modified Bessel function of the second kind of order 3/2 rather than the more standard $K_{3/2}$ to avoid confusion with other $K$'s. The SSA cross section is given by (18), (23), and $G_{1,1}$ as given in the Appendix.

For the results presented, two different values of the rms surface height are used, $h = 0.1$ m and $h = 1$ m, and one value for the correlation length, $l = 10$ m. Frequencies of 100, 300, 1000, and 3000 Hz are used so that the dimensionless parameters $k_{iw} h$ and $k_{iw} l$ range from 0.42 to 12.57 and from 4.19 to 125.66, respectively. For the bistatic scattering strength, the angle of incidence is limited to $45^\circ$. The Biot and other parameters used are those presented by Williams et al. as representative of a sediment seafloor [19].

![Fig. 2. Backscattering strengths for lowest order PT and the lowest order SSA for an rms height of $h = 0.1$ m, a correlation length of $l = 10$ m, and a frequency of $f \equiv 100$ Hz ($k_{iw} h = 0.04$, $k_{iw} l = 4.19$). The roughness spectrum is a modified power law given by (22).](image)

Figs. 2–7 show the results for the backscattering strength. The angles are measured from the positive $z$-axis, and positive incident angles correspond to negative scattering angles (see Fig. 1) for backscattering. In Fig. 2, both the lowest order PT and SSA results are given for a frequency of $f = 100$ Hz, $h = 0.1$ m, and $l = 10$ m ($k_{iw} h = 0.04$, $k_{iw} l = 4.19$). For the surface parameters used, the two approaches are expected to give the same results, which they do. In fact, although not shown, results agree for frequencies up to nearly 1 kHz ($k_{iw} h = 0.42$, $k_{iw} l = 41.89$). In Fig. 3, the surface parameters are the same, but the frequency has increased to 3 kHz ($k_{iw} h = 1.26$, $k_{iw} l = 125.66$). The range of backscattering angles shown is limited to a 20 deg so that differences between the PT and SSA results are easier to see. The PT and SSA results agree for backscattering angles of $-90^\circ$ up to approximately $-75^\circ$ and only start to diverge as the angles approach the normal. Near normal backscattering angles
Fig. 3. Backscattering strengths for lowest order PT and SSA. The surface parameters are the same as in Fig. 2, but the frequency has been increased to \( f = 3 \) kHz \((k_0h = 1.26, k_0l = 125.66)\).

Fig. 4. Backscattering strengths for the lowest order SSA for the same surface parameters as in the previous two figures \((h = 0.1 \text{ m}, l = 10 \text{ m})\) but for four different frequencies.

Fig. 5. Backscattering strengths for lowest order PT and SSA. The surface roughness has been increased with \( h = 1 \text{ m} \) and \( l = 10 \text{ m} \), and the frequency is \( f = 100 \) Hz \((k_0h = 0.42, k_0l = 4.19)\).

Fig. 6. Backscattering strengths for lowest order PT and SSA. The surface parameters are the same as in Fig. 5, but the frequency has been increased to \( f = 3 \) kHz \((k_0h = 12.57, k_0l = 125.66)\).

Fig. 7. Backscattering strengths for the lowest order SSA for the same surface parameters as in the previous two figures \((h = 1 \text{ m}, l = 10 \text{ m})\) but for four different frequencies.

correspond to near specular since for backscattering, an incident angle of \( 0^\circ \) is both the specular and backscattering angle. Since exact results are not available with which to compare the two results, we do not know if either is accurate. However, for other boundary conditions, the lowest order SSA gives good results in the specular direction while the lowest order PT results are inaccurate \([14],[16],[20]\). To see how the SSA backscattering results vary with frequency, in Fig. 4 results are shown for the four different frequencies used in this study. The surface parameters are the same as those used in Fig. 2.

In Figs. 5 and 6, the surface roughness has been increased with \( h = 1 \text{ m} \) and \( l = 10 \text{ m} \). The frequencies are 100 Hz \((k_0h = 0.42, k_0l = 4.19)\) and 3 kHz \((k_0h = 12.57, k_0l = 125.66)\), respectively. In Fig. 5, the PT and SSA results agree fairly closely over most angles, but again differ the most at near normal angles of incidence. In Fig. 6, the PT and SSA results start diverging at an incident angle of approximately \( 45^\circ \) and differ dramatically as the incident angle approaches \( 0^\circ \), differing
Figs. 8–11 show the bistatic scattering strength results as a function of scattering angle for an incident angle of 45°. In Fig. 8 and 9, \( h = 0.1 \) m and \( l = 10 \) m, and the frequencies are \( f = 100 \) Hz \((k_\omega h = 0.04\) and \(k_\omega l = 4.19\)) and 3 kHz \((k_\omega h = 1.26\) and \(k_\omega l = 125.66\)), respectively. As with the backscattering results for these surface parameters, the PT and SSA results agree everywhere for \( f = 100 \) Hz as well as for frequencies up to nearly 1 kHz and differ only in the specular direction for \( f = 3 \) kHz. In Figs. 10 and 11, the surface roughness has been increased with \( h = 1 \) m and \( l = 10 \) m; again the frequencies are \( f = 100 \) Hz \((k_\omega h = 0.42\) and \(k_\omega l = 4.19\)) and 3 kHz \((k_\omega h = 12.57\) and \(k_\omega l = 125.66\)), respectively. For the low frequency case, the PT and SSA results agree well only at grazing angles less than approximately 30°. For the high frequency case, the PT and SSA results agree well only at grazing angles of less than approximately 30° in the back direction. Over a broad range of scattering angles, the PT and SSA results differ significantly.

IV. CONCLUSIONS

In this paper, the lowest order SSA cross section has been derived for scattering from a fluid–fluid-saturated, porous solid (Biot medium) interface. Numerical results for PT and the SSA have been presented and compared for both the backscattering and bistatic scattering strengths for a modified power law spectrum. Frequencies ranging from 100 Hz to 3 kHz have been used for surfaces with \( rms \) heights of 0.1 m and 1 m and a correlation length of 10 m. For backscattering, the PT and SSA results agree for incident grazing angles up to approximately 45° for all surface parameters and frequencies considered (corresponding to...
0.04 \leq k_w h \leq 12.57 \text{ and } 4.19 \leq k_w t \leq 125.66). Thus, there is no advantage to using the SSA for low grazing angle backscatter. However, at high frequencies or for large-scale roughness, the SSA may give more accurate results as the grazing angle is increased. For bistatic scatter, the PT and SSA results agree over all scattering angles for small roughness and at low frequencies. As the surface roughness is increased or the frequency is increased, the results diverge, and in the specular region, the difference is considerable. While it is speculated that the SSA results are more accurate, exact results are needed for comparison for the accuracy to be determined. However, earlier studies indicate that when the SSA and PT results agree, the PT results are accurate.

APPENDIX

Equations (6) and (7) define two terms $G_{0,1}$ and $G_{1,1}$ which are the first terms of the column vectors $G_0$ and $G_1$, respectively

\begin{equation}
G_{0,1} = [G_0]_1
\end{equation}

\begin{equation}
G_{1,1} = [G_1]_1
\end{equation}

where the column vectors $G_0$ and $G_1$ are given by

\begin{equation}
G_0 = \frac{k_w}{k_0} Y_1(K_s) S_0(K_s)
\end{equation}

\begin{equation}
G_1 = \frac{i k_w}{k_0} Y_1(K_s) \left[ P^{(3)}(K_s) \right]^{-1} B(K_s, K_i) D_0(K_i).
\end{equation}

In this appendix, the matrices and vectors of (26) and (27) are given. They are virtually identical to the ones given in [9] and are presented here for completeness and convenience. First

\begin{equation}
Y_1(K) = \begin{bmatrix}
\nu_{0w}(K) & 0 & 0 & 0 & 0 \\
0 & \nu_{1w}(K) & 0 & 0 & 0 \\
0 & 0 & \nu_{2w}(K) & 0 & 0 \\
0 & 0 & 0 & \nu_{4w}(K) & 0 \\
0 & 0 & 0 & 0 & \nu_{4w}(K)
\end{bmatrix}
\end{equation}

where

\begin{equation}
\nu_{\alpha}(K) = \sqrt{\alpha_w^2 - K^2 / k_0^2}, \quad \alpha = w, 1, 2, t
\end{equation}

are the normalized vertical wavenumbers with the subscript $\alpha$ representing the compressional wave in the upper fluid ($w$), the fast compressional wave in the Biot medium (1), the slow compressional wave (2), and the shear wave (t) and where

\begin{equation}
\alpha_w = \frac{c_w}{c_w}
\end{equation}

are the ratios of the sound speeds with the sound speed in the upper fluid. As mentioned earlier, $S_0$ is the 5 \times 1 column vector whose elements are the reflection and transmission coefficients for a flat interface

\begin{equation}
S_0(K_i) = \begin{bmatrix}
W_w(K_i) \\
W_1(K_i) \\
W_2(K_i) \\
W_v(K_i) \\
W_h(K_i)
\end{bmatrix}
\end{equation}

where the elements are the reflection coefficient and four transmission coefficients for a plane wave incident from the upper fluid. These elements are found using

\begin{equation}
S_0(K_i) = - \left[ P^{(3)}(K_i) \right]^{-1} Q^{(3)}(K_i)
\end{equation}

where $P^{(3)}$ is one of the three 5 \times 5 matrices $P^{(n)}(K)$, $n = 1, 2, 3$, with elements given by

\begin{equation}
P_{wn}^{(3)}(K) = K_w k_w \delta_{mn}, \quad m = 1, 2, 3
\end{equation}

\begin{equation}
P_{w1}^{(3)}(K) = - k_w k_w \delta_{3n}
\end{equation}

\begin{equation}
P_{w2}^{(3)}(K) = - c_{w1} k_w \delta_{3n}
\end{equation}

\begin{equation}
P_{1m}^{(3)}(K) = - \frac{k_2^2}{k_w} \left[ (H - 2 \mu - \gamma_2 C) \delta_{mn} + 2 \mu_1 c_{1m}(K) c_{1n}(K) \right],
\end{equation}

\begin{equation}
P_{14}^{(3)}(K) = - \frac{k_2^2}{k_w} (M - C) \delta_{n3}
\end{equation}

\begin{equation}
P_{25}^{(3)}(K) = \frac{k_1}{k_w} (1 - \gamma_1) c_{2n}(K)
\end{equation}

\begin{equation}
P_{2m}^{(3)}(K) = - \frac{k_2^2}{k_w} \left[ (H - 2 \mu - \gamma_2 C) \delta_{mn} + 2 \mu_2 c_{2m}(K) c_{2n}(K) \right],
\end{equation}

\begin{equation}
P_{24}^{(3)}(K) = - \frac{k_2^2}{k_w} (M - C) \delta_{n3}
\end{equation}

\begin{equation}
P_{25}^{(3)}(K) = \frac{k_2}{k_w} (1 - \gamma_2) c_{2n}(K)
\end{equation}

\begin{equation}
P_{wm}^{(3)}(K) = \frac{\mu k_0}{k_w} \left[ c_{wm}(K) c_{wn}(K) + c_{wm}(K) c_{wn}(K) \right],
\end{equation}

\begin{equation}
P_{w1}^{(3)}(K) = 0
\end{equation}

\begin{equation}
P_{w2}^{(3)}(K) = - \frac{k_2}{k_w} (1 - \gamma_2) c_{2n}(K)
\end{equation}

\begin{equation}
P_{hm}^{(3)}(K) = - \frac{\mu k_0}{k_w} \left[ c_{hm}(K) c_{hn}(K) + c_{hm}(K) c_{hn}(K) \right],
\end{equation}

\begin{equation}
P_{h4}^{(3)}(K) = 0
\end{equation}

\begin{equation}
P_{h5}^{(3)}(K) = \frac{k_1}{k_w} (1 - \gamma_1) c_{hn}(K)
\end{equation}

where $K_w$ is the bulk modulus of the upper fluid, and $Q^{(3)}$ is one of the three $5 \times 1$ column vectors $Q^{(n)}(K)$, $n = 1, 2, 3$, with elements given by

\begin{equation}
Q_{wm}^{(3)}(K) = K_w k_w \delta_{mn}, \quad m = 1, 2, 3
\end{equation}

\begin{equation}
Q_{h4}^{(3)}(K) = - k_w k_w \delta_{n3}
\end{equation}

\begin{equation}
Q_{h5}^{(3)}(K) = - c_{hn}(K).
\end{equation}

There are three $5 \times 6$ matrices $E^{(n)}$, $n = 1, 2, 3$, that are a combination of $P^{(n)}$ and $Q^{(3)}$ as shown in (51) at the bottom of the next page, where the rows correspond to the five boundary conditions, the first five columns represent the scattered field.
and the four transmitted fields, respectively, and the last column represents the incident field. In (27)

\[ B(K_s, K_i) = \frac{1}{\mu_0} \left[ (K_{s1} - K_{i1}) E^{(1)}(K_i) + (K_{s2} - K_{i2}) E^{(2)}(K_i) - E^{(3)}(K_i) Y_2(K_i) \right] \]

and the 6 × 1 column vector \( D_0 \) is formed by concatenating \( S_0 \) and 1

\[ D_0(K_i) = \begin{bmatrix} S_0(K_i) \\ \vdots \\ 1 \end{bmatrix}. \]  

In (52), \( K_{i3} \) and \( K_{s3} \), \( \beta = 1, 2 \), are the magnitudes of the 2-D incident and scattered wave vectors, respectively, for the fast and slow compressional waves, and \( Y_2 \) is the 6 × 6 matrix shown in (54) at the bottom of the page with the nonzero elements given by (29) and (30).

A number of quantities must be defined in the equations given above. The Kronecker delta function \( \delta_{mn} \) is equal to zero except when \( m = n \), the \( e^{\pm} \) functions are the scalar products of the unit vectors specifying the directions of propagation of the different field components and the three coordinate directions \( (x_1, x_2, x_3) \). The unit vectors are given by

\[ e^{\pm}_\alpha(K) = \frac{k^{\pm}_\alpha(K)}{k_\alpha}, \quad \alpha = w, 1, 2, t \]

where

\[ k^{\pm}_\alpha(K) = (K, \pm k_\alpha \nu_\alpha(K)), \quad \alpha = w, 1, 2, t \]

are the wave vectors, with the \( \pm \) superscript indicating whether the wave is upward going (+) or downward going (−), for the upper fluid compressional wave \( (w) \), the fast compressional wave \( (1) \), the slow compressional wave \( (2) \), and the shear wave \( (t) \), and

\[ k_\alpha = \frac{\omega}{c_\alpha}, \quad \alpha = w, 1, 2, t \]

are the wavenumbers. Unit vectors for the shear wave can be defined for the horizontal and vertical polarizations which are transverse to the direction of propagation \( \hat{e}_t \). These unit vectors give the direction of particle movement

\[ e^{\pm}_n(K) = \frac{e^{\pm}_t \times k^{\pm}_n(K)}{k_t} = \pm \frac{K_\alpha \nu_\alpha(K)}{K} + \hat{x}_3 \frac{K}{k_t} \]

Thus

\[ e^{\pm}_n = e^{\pm}_\alpha \cdot \hat{x}_n, \quad n = 1, 2, 3; \quad \alpha = w, 1, 2, t, v, h. \]

Other parameters include those related to the Biot medium

\[ H = (K_r - K_b) \frac{\mu}{D - K_b} + K_b + \frac{4\mu}{3} \]

\[ C = K_r \frac{K_r - K_b}{D - K_b} \]

\[ M = -K_b \frac{K_r^2}{D - K_b} \]

\[ \rho = \beta \rho_f + (1 - \beta) \rho_s \]

\[ D = K_r \left[ 1 + \beta \left( \frac{K_r}{K_f} - 1 \right) \right] \]

where \( K_r \) is the sediment grain bulk modulus, \( K_b \) is the frame bulk modulus, \( \mu \) is the frame shear modulus, \( \beta \) is the sediment
porosity, $\rho_f$ is the interstitial fluid density, $\rho_s$ is the sediment grain density, and $K_f$ is the interstitial-fluid modulus. Also

$$\gamma_2 = \frac{\rho_f^2 \omega^2 - \rho_f c_s^2 - \rho_s c_p^2}{\rho_f c_s^2 - \rho_s c_p^2}, \quad \beta = 1, 2 \tag{66}$$

$$\gamma_1 = \frac{\rho_f - \mu}{\rho_f c_s^2}, \quad \eta = 0 \tag{67}$$

In (57), the wavenumber for the upper fluid compressional wave is real, but the other three wavenumbers are complex and, thus, include loss. The wavenumbers for the fast and slow waves are found using

$$\left(H k_\beta^2 - \rho_\omega^2 \right) \left(\frac{\alpha \rho_\omega^2}{\beta} - M k_\beta^2 + i \frac{F \eta \omega}{\kappa} \right) + (C k_\beta^2 - \rho_\omega^2)^2 = 0, \quad \beta = 1, 2 \tag{68}$$

and for the shear wave

$$\left(\mu k_\beta^2 - \rho_\omega^2 \right) \left(\frac{\alpha \rho_\omega^2}{\beta} + i \frac{F \eta \omega}{\kappa} \right) + \rho^2 \omega^4 = 0 \tag{69}$$

where $\omega$ is the radian frequency, $\alpha$ is the tortuosity, $\eta$ is the interstitial fluid viscosity, $\kappa$ is the permeability, and $F$ is the dissipation function which is modeled by

$$F = \frac{\varepsilon T(\varepsilon)}{1 - \frac{\varepsilon^2}{\varepsilon} T(\varepsilon)} \tag{70}$$

where

$$T = -\sqrt{i} J_1 \left(\varepsilon \sqrt{i} \right) / J_0 \left(\varepsilon \sqrt{i} \right) \tag{71}$$

$$\varepsilon = a \sqrt{\frac{\omega \rho_f}{\eta}} \tag{72}$$

$J_0$ and $J_1$ are Bessel functions of the first kind, and $a$ is the pore size.

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REFERENCES


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