1. **Text problem 4.1**

By direct substitution of \( f(x) \) [Eq. (4.2-6)] into \( F(u) \) [Eq. (4.2-5)]:

\[
F(u) = \frac{1}{M} \sum_{x=0}^{M-1} \left[ \sum_{r=0}^{M-1} F(r) e^{j2\pi rx/M} \right] e^{-j2\pi ux/M}
\]

\[
= \frac{1}{M} \sum_{r=0}^{M-1} F(r) \sum_{x=0}^{M-1} e^{j2\pi rx/M} e^{-j2\pi ux/M}
\]

\[
= \frac{1}{M} F(u) [M] = F(u)
\]

where the third step follows from the orthogonality condition given in the problem statement. Substitution of \( F(u) \) into \( f(x) \) is handled in a similar manner.

2. **Text problem 4.2**

This is a simple problem to familiarize the student with just the manipulation of the 2-D Fourier transform and its inverse. The Fourier transform is linear iff:

\[
\Re [a_1 f_1(x, y) + a_2 f_2(x, y)] = a_1 \Re [f_1(x, y)] + a_2 \Re [f_2(x, y)]
\]

where \( a_1 \) and \( a_2 \) are arbitrary constants. From the definition of the 2-D transform,

\[
\Re [a_1 f_1(x, y) + a_2 f_2(x, y)] = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [a_1 f_1(x, y) + a_2 f_2(x, y)] e^{-j2\pi (ux/M + vy/N)}
\]

\[
= \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} a_1 f_1(x, y) e^{-j2\pi (ux/M + vy/N)}
\]

\[
+ \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} a_2 f_2(x, y) e^{-j2\pi (ux/M + vy/N)}
\]

\[
= a_1 \Re [f_1(x, y)] + a_2 \Re [f_2(x, y)]
\]

which proves linearity. The inverse is done in the same way.

3. **Text problem 4.4**

An important aspect of this problem is to recognize that the quantity \((u^2 + v^2)\) can be replaced by the distance squared, \(D^2(u, v)\). This reduces the problem to one variable, which is notationally easier to manage.
Rather than carry an award capital letter throughout the development, we define \( w^2 \equiv D^2(u, v) = (u^2 + v^2) \). Then we proceed as follows:

\[
H(w) = e^{-w^2/2\sigma^2}.
\]

The inverse Fourier transform is

\[
h(z) = \int_{-\infty}^{\infty} H(w) e^{j2\pi w z} dw
\]

\[
= \int_{-\infty}^{\infty} e^{-w^2/2\sigma^2} e^{j2\pi w z} dw
\]

\[
= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [w^2 - j4\pi \sigma^2 wz]} dw.
\]

We now make use of the identity

\[
e^{-\frac{(2\pi)^2 z^2 \sigma^2}{2}} e^{-\frac{(2\pi)^2 z^2 \sigma^2}{2}} = 1.
\]

Inserting this identity in the preceding integral yields

\[
h(z) = e^{-\frac{(2\pi)^2 z^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [w^2 - j4\pi \sigma^2 wz - (2\pi)^2 z^2]} dw
\]

\[
= e^{-\frac{(2\pi)^2 z^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [w - j2\pi \sigma^2 z]^2} dw.
\]

Next we make the change of variable \( r = w - j2\pi \sigma^2 z \). Then, \( dr = dw \) and the above integral becomes

\[
h(z) = e^{-\frac{(2\pi)^2 z^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr.
\]

Finally, we multiply and divide the right side of this equation by \( \sqrt{2\pi} \sigma \):

\[
h(z) = \sqrt{2\pi} \sigma e^{-\frac{(2\pi)^2 z^2 \sigma^2}{2}} \left[ \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr \right].
\]

The expression inside the brackets is recognized as a Gaussian probability density function, whose integral from \( -\infty \) to \( \infty \) is 1. Then

\[
h(z) = \sqrt{2\pi} \sigma e^{-\frac{(2\pi)^2 z^2 \sigma^2}{2}}.
\]

Going back to two spatial variables gives the final result:

\[
h(x, y) = \sqrt{2\pi} \sigma e^{-2\pi^2 \sigma^2 (x^2 + y^2)}.
\]

4. **Text problem 4.5**

The spatial filter is obtained by taking the inverse Fourier transform of the frequency-domain filter:

\[
h_{lp}(x, y) = \mathcal{F}^{-1} \left[ 1 - H_{lp}(u, v) \right]
\]

\[
= \mathcal{F}^{-1} \left[ 1 \right] - \mathcal{F}^{-1} \left[ H_{lp}(u, v) \right]
\]

\[
= \delta(0) - \sqrt{2\pi} \sigma e^{-2\pi^2 \sigma^2 (x^2 + y^2)}
\]
5. **Text problem 4.14**

(a) The spatial average is

\[
g(x, y) = \frac{1}{4} [f(x, y + 1) + f(x + 1, y) + f(x - 1, y) + f(x, y - 1)].
\]

From Eq. (4.6-2),

\[
G(u, v) = \frac{1}{4} \left[ e^{j2\pi v/N} + e^{j2\pi u/M} + e^{-j2\pi u/M} + e^{-j2\pi v/N} \right] F(u, v)
\]

\[
= H(u, v) F(u, v),
\]

where

\[
H(u, v) = \frac{1}{2} \left[ \cos(2\pi u/M) + \cos(2\pi v/N) \right]
\]

is the filter transfer function in the frequency domain.

(b) To see that this is a lowpass filter, it helps to express the preceding equation in the form of our familiar centered functions:

\[
H(u, v) = \frac{1}{2} \left[ \cos(2\pi [u - M/2]/M) + \cos(2\pi [v - N/2]/N) \right].
\]

Consider one variable for convenience. As \( u \) ranges from 0 to \( M \), the value of \( \cos(2\pi [u - M/2]/M) \) starts at \(-1\), peaks at \( 1 \) when \( u = M/2 \) (the center of the filter) and then decreases to \(-1\) again when \( u = M \). Thus, we see that the amplitude of the filter decreases as a function of distance from the origin of the centered filter, which is the characteristic of a lowpass filter. A similar argument is easily carried out when considering both variables simultaneously.

6. **Text problem 4.15**

The problem statement gives the form of the difference in the \( x \)-direction. A similar expression gives the difference in the \( y \)-direction. The filtered function in the spatial domain then is:

\[
g(x, y) = f(x, y) - f(x + 1, y) + f(x, y) - f(x, y + 1).
\]

From Eq. (4.6-2),

\[
G(u, v) = F(u, v) - F(u, v)e^{j2\pi u/M} + F(u, v) - F(u, v)e^{j2\pi v/N}
\]

\[
= [1 - e^{j2\pi u/M}] F(u, v) + [1 - e^{j2\pi v/N}] F(u, v)
\]

\[
= H(u, v) F(u, v),
\]

where \( H(u, v) \) is the filter function:

\[
H(u, v) = -2j \left[ \sin(\pi u/M)e^{j\pi v/M} + \sin(\pi v/N)e^{j\pi u/N} \right].
\]

(b) To see that this is a highpass filter, it helps to express the filter function in the from of our familiar centered functions:
Consider one variable for convenience. As $u$ ranges from 0 to $M$, $H(u, v)$ starts at its maximum (complex) value of $2j$ for $u = 0$ and decreases from there. When $u = M/2$ (the center of the shifted function), a similar argument is easily carried out when considering both variables simultaneously. The value of $H(u, v)$ starts increasing again and achieves the maximum value of $2j$ again when $u = M$. Thus, this filter has a value of 0 at the origin and increases with increasing distance from the origin. This is the characteristic of a highpass filter. A similar argument is easily carried out when considering both variables simultaneously.

7. **Text problem 4.17**

(a) Express filtering as convolution to reduce all processes to the spatial domain. Then, the filtered image is given by

$$g(x, y) = h(x, y) * f(x, y)$$

where $h$ is the spatial filter (inverse Fourier transform of the frequency-domain filter) and $f$ is the input image. Histogram processing this result yields

$$g'(x, y) = T[g(x, y)] = T[h(x, y) * f(x, y)],$$

where $T$ denotes the histogram equalization transformation. If we histogram-equalize first, then

$$g(x, y) = T[f(x, y)]$$

and

$$g'(x, y) = h(x, y) * T[f(x, y)]$$

In general, $T$ is a nonlinear function determined by the nature of the pixels in the image from which it is computed. Thus in general, $T[h(x, y) * f(x, y)] \neq h(x, y) * T[f(x, y)]$ and the order does matter.

(b) As indicated in Section 4.4 highpass filtering severely diminishes the contrast of an image. Although high-frequency emphasis helps some, the improvement is usually not dramatic (see Fig. 4.30). Thus, if an image is histogram equalized first, the gain in contrast improvement will essentially be lost in the filtering process. Therefore, the procedure in general is to filter first and histogram-equalize the image after that.

8. **Text problem 4.21**
Recall that the reason for padding is to establish a “buffer” between the periods that are implicit in the DFT. Imagine the image on the left being duplicated infinitely many times to cover the $xy$-plane. The result would be a checkerboard, with each square being in the checkerboard being the image (and the black extensions). Now imagine doing the same thing to the image on the right. The results would be indistinguishable. Thus, either form of padding accomplishes the same separation between images, as desired.

9. **Text problem 4.27**

The problem can be solved by carrying out the following steps:

1. Perform a median filtering operation.
2. Follow (1) by high-frequency emphasis.
3. Histogram-equalize this result.
4. Compute the average gray level, $K_0$. Add the quantity $(K - K_0)$ to all pixels.
5. Perform the transformations shown in Fig. P4.27, where $r$ is the input gray level, and $R$, $G$, and $B$ are fed into an RGB color monitor.

![Figure P4.27](image)

10.

We will consider the 1-D case for simplicity. Suppose $X(u)$ is the given sequence, for which we wish to compute the inverse DFT. In other words, we need to find

$$x(m) = \frac{1}{M} \sum_{m=0}^{M-1} X(u) e^{j2\pi \frac{m}{M}}.$$ 

Consider the DFT of $X(u)$, that is, input $X(u)$ to the given DFT routine to get an output $y(m)$:
\[
y(m) = \sum_{m=0}^{M-1} X(u)e^{-j2\pi \frac{mu}{M}}
\]

\[
\frac{1}{M} y(m) = \frac{1}{M} \sum_{m=0}^{M-1} X(u)e^{j2\pi \frac{(-u+m)}{M}}
\]

Setting \( m = 0 \), we can see that \( \frac{1}{M} y(0) = x(0) \). For \( m = 1, \ldots, M - 1 \), consider:

\[
\frac{1}{M} y(M-m) = \frac{1}{M} \sum_{m=0}^{M-1} X(u)e^{j2\pi \frac{(-m-(M-m))}{M}} = \frac{1}{M} \sum_{m=0}^{M-1} X(u)e^{j2\pi \frac{mu}{M}} e^{j2\pi \frac{-mu}{M}}
\]

\[
= \frac{1}{M} \sum_{m=0}^{M-1} X(u)e^{j2\pi \frac{mu}{M}} e^{-j2\pi u} = \frac{1}{M} \sum_{m=0}^{M-1} X(u)e^{j2\pi \frac{mu}{M}} = x(m)
\]

In the above, we have used the fact that \( e^{j2\pi u} = \cos(2\pi u) + j\sin(2\pi u) = 1 \), whenever \( u \) is an integer. Therefore sequence \( x(m) \) can be easily obtained from sequence \( y(m) \), which in turn is obtained by the DFT routine provided.