External Equivalent

EE 521 Analysis of Power Systems

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EXTERNAL EQUIVALENT

Each power system (area) is part of an interconnected system. Monitoring devices are installed and data are available from one’s own system. However, security analysis depends on accurate and reliable load flow models of the non-monitored parts of the system as well. Approximate reduced models for load flow studies of the non-monitored parts are called External Equivalents.
1. Ward Equivalent (=1948)
   \[
   \begin{cases}
   i : \text{internal} \\
   b : \text{boundary} \\
   e : \text{external}
   \end{cases}
   \]

By nodal analysis. (YE = I)

\[
\begin{array}{ccc}
  & e & b & i \\
 e & \overline{Y}_{ee} & \overline{Y}_{eb} & 0 \\
 b & \overline{Y}_{be} & \overline{Y}_{bb} + \overline{Y}_{bi} & \overline{Y}_{bi} \\
 i & 0 & \overline{Y}_{ib} & \overline{Y}_{ii}
\end{array}
\]

\[
\begin{align*}
\overline{Y}_{ee} \overline{E}_e + \overline{Y}_{eb} \overline{E}_b &= \overline{I}_e \\
\Rightarrow \overline{E}_e &= - \overline{Y}_{ee}^{-1} \overline{Y}_{eb} \overline{E}_b + \overline{Y}_{ee}^{-1} \overline{I}_e
\end{align*}
\]
By (2):
\[ \overline{Y}_{be} \overline{E}_e + (\overline{Y}_{bb} + \overline{Y}_{bi}) \overline{E}_b + \overline{Y}_{bi} \overline{E}_i = \overline{I}_b \]
\[ \implies - \overline{Y}_{be} \overline{Y}_{ee}^{-1} \overline{Y}_{eb} \overline{E}_b + \overline{Y}_{be} \overline{Y}_{ee} \overline{I}_e \]
\[ + (\overline{Y}_{bb} + \overline{Y}_{bi}) \overline{E}_b + \overline{Y}_{bi} \overline{E}_i = \overline{I}_b \]
\[ \Rightarrow [\overline{Y}_{bb} + \overline{Y}_{bi} - \overline{Y}_{be} \overline{Y}_{ee}^{-1} \overline{Y}_{eb}] \overline{E}_b + \overline{Y}_{bi} \overline{E}_i \]
\[ = \overline{I}_b - \overline{Y}_{be} \overline{Y}_{ee}^{-1} \overline{I}_e \]

Let : \[ \overline{Y}_{eq} = \overline{Y}_{bb} - \overline{Y}_{be} \overline{Y}_{ee}^{-1} \overline{Y}_{eb} \]
\[ \overline{I}_{eq} = - \overline{Y}_{be} \overline{Y}_{ee}^{-1} \overline{I}_e \]

\[
\begin{array}{c|c}
\overline{Y}_{bb} + \overline{Y}_{eq} & \overline{Y}_{bi} \\
\hline
\overline{Y}_{ib} & \overline{Y}_{ii}
\end{array}
\begin{array}{c|c}
\overline{E}_b & \overline{E}_i
\end{array}
= \begin{array}{c}
\overline{I}_b + \overline{I}_{eq}
\end{array}
\begin{array}{c}
\overline{I}_i
\end{array}
\]

(4)
In block diagram:

<table>
<thead>
<tr>
<th>Internal System (i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{Y}_{bb}$</td>
</tr>
<tr>
<td>$\bar{Y}_{ib}$</td>
</tr>
</tbody>
</table>

Ward Equivalent

**Four Basic Steps** in the construction of a ward Equivalent:

1. Determination of the external network data from available information
2. Obtaining the ward equivalent network $\bar{Y}_{eq}$ by Gaussian elimination.
3. Using the values for the complex voltages at the boundary buses from the internal state-estimation to compute the flows in the Ward equivalent branches.

4. Boundary matching, i.e., adding fictitious injections at the boundary buses so that Eq.(4) holds.

**Base case:** results of the state estimation (load flow) for the internal system without contingency.
Why are the original and reduced models equivalent?

In the base case, if one solves the load flow (4), the solution \([ E_b, E_i ]\) (system state) is exactly the same as that of Eqs. (1)-(3). The external equivalent model can be used in the contingency evaluation.
• Ward Equivalent gives reasonably accurate results for real power flows, whereas the accuracy for reactive power flow is relatively poor.

(This is due to the fact that the change in reactive power injection to maintain constant voltage at external PV buses is not accounted for.)
2. Ward-PV Equivalent

- The Ward reduction process is applied only to external PQ buses.
- The external PV buses are retained.
- The Ward-PV equivalents give excellent results for contingency evaluation.

Given the system block diagram:

![System block diagram](image)
(Node (Bus) Admittance Matrix $Y$)
To obtain the ward-PV equivalent, one needs to do Gaussian elimination to eliminate Q part, i.e.,

$$
\begin{array}{ccc}
Q & V & b \\
\hline
Q & \overline{Y}^w_{vv} & \overline{Y}^w_{vb} & 0 \\
V & 0 & \overline{Y}^w_{bv} & \overline{Y}_{bb} + \overline{Y}_{vb} & \overline{Y}_{bi} \\
b & 0 & \overline{Y}_{ib} & \overline{Y}_{ii} \\
i & 0 & 0 & 0 & 0
\end{array}
$$

(+) 

The Ward-PV equivalent network:
3. Extended Ward

- To combine the simplicity of the Ward equivalent with the response of the Ward-PV equivalent.
- It is a Ward equivalent with additional reactive support at the boundary buses such that its reactive response is close to that of the Ward-PV equivalent.
- The reactive support in the extended Ward is derived so that the incremental response (linearized response from the base case) for the reactive power flows is almost the same as that from the Ward-PV equivalent.
Incremental form of Decoupled Load Flow:

\[
\begin{align*}
\left( \frac{\Delta P}{V} \right) &= B ' \Delta \theta, \\
\left( \frac{\Delta Q}{V} \right) &= B ' ' \Delta V,
\end{align*}
\]

Consider the decoupled load flow:

\[
\begin{align*}
P_i &= \sum_{j \neq i} V_i V_j B_{ij} (\theta_i - \theta_j) \\
Q_i &= -V_i^2 B_{ii} - \sum_{j \neq i} V_i V_j B_{ij}
\end{align*}
\]

Incremental form:

\[
\begin{align*}
\Delta P_i &= \sum_{j \neq i} V_i V_j B_{ij} (\Delta \theta_i - \Delta \theta_j) \\
\text{or : } \frac{\Delta P_i}{V_i} &= \sum_{j \neq i} V_j B_{ij} (\Delta \theta_i - \Delta \theta_j) \\
\text{Set } V_j = 1 \text{, then : }
\end{align*}
\]

\[
\left( \frac{\Delta P}{V} \right) = B ' \Delta \theta \quad \text{(DC load flow)}
\]
On the other hand:

\[
\begin{align*}
\frac{\partial Q_i}{\partial V_i} &= -2V_iB_i - \sum_{j \neq i} V_j B_j \\
\frac{\partial Q_j}{\partial V_j} &= -V_j B_j \quad (i \neq j)
\end{align*}
\]

so:

\[
\begin{align*}
V_i^{-1}\frac{\partial Q_i}{\partial V_i} &= -2 \quad B_i - \sum_{j \neq i} \frac{V_j}{V_i} B_j \\
V_j^{-1}\frac{\partial Q_j}{\partial V_j} &= -B_j \quad (i \neq j)
\end{align*}
\]

As \( \Delta Q = \frac{\partial Q}{\partial V} \Delta V \), where \( \frac{\partial Q}{\partial V} = \left[ \frac{\partial Q}{\partial V_1} \frac{\partial Q}{\partial V_2} \ldots \right] \)

\[
\begin{bmatrix}
V_1^{-1} & 0 \\
V_2^{-1} & 0 \\
\vdots & \ddots
\end{bmatrix} \Delta Q = V^{-1} \frac{\partial Q}{\partial V} \Delta V
\]

\[
\left( \frac{\Delta Q}{V} \right) = V^{-1} \frac{\partial Q}{\partial V} \Delta V = B^{'''} \Delta V
\]

For \( B^{'''} \): \( B_i = -2B_i - \sum_{j \neq i} B_j \)

\[
\overline{Y}_{ii} = \frac{G_i}{|jB_i|} = 2 \left[ \sum_{j \neq i} B_j + \frac{B_{i0}}{B_{i0}} \right] - \sum_{j \neq i} B_j
\]

\[
= \sum_{j \neq i} B_j + \frac{2}{B_{i0}} B_{i0}
\]
\[ B''_{ij} = -B_{ij} \quad (i \neq j) \]

B” can be obtained from B’ by deleting rows & columns corresponding to PV buses, doubling the shunts.

\[
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\]

From Eq. (+), if the rows and columns corresponding to the external PV buses are deleted, then the decoupled reactive power flow is given by:

\[
\begin{bmatrix}
B''_{WV} + (B_{bb})^i : B''_{bi} \\
\vdots \\
B''_{ib} : B''_{ii}
\end{bmatrix}
\begin{bmatrix}
\Delta V_{-b} \\
\vdots \\
\Delta V_{-i}
\end{bmatrix} =
\begin{bmatrix}
\Delta Q_{-b} \\
\vdots \\
\Delta Q_{-i}
\end{bmatrix}
\]

\[
\frac{\Delta V_{-b}}{V_{-b}}
\]

\[
\frac{\Delta V_{-i}}{V_{-i}}
\]
The reactive response of the Ward-PV Equivalent to the changes in boundary bus voltage $\Delta V_b$ is

$$\Delta Q_{\text{W}_b} = [V_b] B_{\text{W}_b} \Delta V_b$$

$$[V_b] = \begin{bmatrix} V_{b_1} & 0 \\ 0 & \ddots \end{bmatrix}$$  \hfill (*)

On the other hand if one starts with the original networks and performs the Ward Equivalent, the decoupled reactive power flow would be

$$\begin{bmatrix} B'' + \left( B_{bb}^i \right)' & B''_{bi} \\ B_{ib}' & B''_{ii} \end{bmatrix} \begin{bmatrix} \Delta V_b \\ \Delta V_i \end{bmatrix} = \begin{bmatrix} \frac{\Delta Q_b}{V_b} \\ \frac{\Delta Q_i}{V_i} \end{bmatrix}$$

The reactive response of the Ward equivalent is given by:

$$\Delta Q_b^W = [V_b] B'' \Delta V_b$$  \hfill (\Delta)
By equations (*) & (\(\Delta\)). If the Ward Equivalent is desired to have the same reactive response as the Ward-PV equivalent, reactive injections should be of the amount:

\[
\Delta Q \quad \sim \quad \Delta V_b \quad \left[ V_b \right] \left( B''_{Wv} - B''_W \right) \Delta V_b
\]

An approximation to (#) will be derived that can be implemented in a power flow program. The approach is to construct a network the corresponding B’’ matrix of which is \(\hat{B}'\)
Consider the k-th component of $\Delta Q_b$, i.e.,

$$
\frac{\Delta Q_k}{V_k} = B_{kk} \Delta V_k + \sum_{m \neq k} B_{km} \Delta V_m
$$

$$
= B_k \Delta V_k + \sum_{m \neq k} B_{km} (\Delta V_m - \Delta V_k)
$$

Where: $B_k = B_{kk} + \sum_{m \neq k} B_{km}$

In other words, the corresponding network has a shunt given by $B_k$. 
Now suppose

\[ B''_{wv} = \begin{bmatrix} B_{km} \end{bmatrix} \quad \text{and} \quad B''_w = \begin{bmatrix} B_{km} \end{bmatrix} \]

Then:

\[ B''_{kk} = - \sum_{m \neq k} B_{km} - \sum_{j \in v} B_{kj} - B_k \quad \text{(shunt)} \]

\[ B''_{kk} = - \sum_{m \neq k} B_{km} - \begin{bmatrix} B_k \end{bmatrix} \quad \text{(shunt)} \]

Therefore:

\[ \overset{\wedge}{B}_{kk} = B''_{kk} - B''_{kk} \]

\[ = - \sum_{j \in v} B_{kj} \]

Practically:

\[ \Delta V_m \approx \Delta V_k \quad \text{and} \quad B''_{wv} \approx B''_w, \quad \text{so:} \]

\[ \overset{\wedge}{B}_k \approx \overset{\wedge}{B}_{kk} = - \sum_{j \in v} B_{kj} \]
By the above derivation, an approximate formula is obtained:

\[ \Delta \tilde{Q}_k = V_k \hat{B}_k \Delta V_k \]
The construction of an extended Ward equivalent is summarized below:

1. Obtain a Ward Equivalent of the external system.

2. Start again from the original system. Ground all external PV buses.

Apply Gaussian Elimination on the Bus Admittance Matrix $Y$ to eliminate all external buses to obtain the equivalent shunts at the boundary buses, which are the admittances $\hat{j} B_k$. 
3. Augment the Ward Equivalent by inserting a shunt $j \frac{B^k}{2}$ at each boundary bus.

- The extended Ward equivalent has been found to give accurate results for contingency evaluation.
Further Information