Steady-State Security Region Concepts

EE 521 Analysis of Power Systems

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STEADY-STATE SECURITY REGIONS

DC load flow

\[ B\theta = P \quad (\theta = XP) \]

Line flow constraints:

\[ -\delta \leq A^T\theta \leq \delta \]

Recall that

\[ \{\theta : -\delta \leq A^T\theta \leq \delta\} \]

\[ \{P : -\delta \leq A^T\theta \leq \delta, B\theta = P\} \]
Steady-state security region

\[ \mathcal{R}_p = \{ \mathbf{P} : -\delta \leq \mathbf{A}^T \mathbf{\theta} \leq \delta \} \]

It is desirable to obtain a region of the form

\[ \mathcal{R} = \{ \mathbf{P} : \mathbf{P}^m \leq \mathbf{P} \leq \mathbf{P}^M \} \subset \mathcal{R}_p \]

The advantages are:

1) \( \mathcal{R} \) has a very explicit and simple description
2) If \( \mathbf{P} \in \mathcal{R} \), then it is guaranteed that the line flow constraints will be satisfied.
3) For on-line operation, if \( \mathcal{R} \) is obtained (by off-line computation) and stored in the computer, then it can be used in the contingency evaluation.
Security Region
\[ R_p = \{ \mathbf{P} : \mathbf{P} = B \theta, -\delta \leq A^T \theta \leq \delta, \: G^m \leq G \leq G^M \} \]

Hyperbox Approximation
\[ D_p = \{ \mathbf{P} : \mathbf{P}^m \leq \mathbf{P} \leq \mathbf{P}^M \} \]
We will derive a steady-state security region in the following. Consider the line flow constraints

$$-\delta_{ij} \leq \theta_i - \theta_j \leq \delta_{ij}$$

Note that if $|\theta_i| < \delta_{ij}/2$ and $|\theta_j| < \delta_{ij}/2$, then

$$|\theta_i - \theta_j| \leq |\theta_i| + |\theta_j| < \delta_{ij}$$

Therefore, for all lines in the system, if

$$|\theta| < \frac{1}{2} \cdot \text{Min}_{(i,j)} \delta_{ij}$$

for all nodes $i$

then the line flow constraints will be satisfied.

Now consider the DC load flow

$$\underline{\theta} = XP$$

The i-th component

$$\theta_i = \sum_{j=1}^{n-1} X_{ij} P_j$$

(n: number of nodes)
\[ |\theta_i| \leq \sum_{j=1}^{n-1} |X_{ij}| |P_j| \]

\[ = \sum_{j=1}^{n-1} X_{ij} |P_j| \quad (X_{ij} \geq 0) \]

\[ \leq \sum_{j=1}^{n-1} X_{ij} |P_j| \quad (X_{ii} \geq X_{ij}) \]

Therefore, if for every node j,

\[ |P_j| \leq \frac{1}{(n-1)X_{jj}} \cdot \frac{1}{2} \cdot \min_{(i,j)} \delta_{ij} \quad j = 1, \ldots, n-1 \]

Then:

\[ |\theta_i| \leq \sum_{j=1}^{n-1} |X_{jj}| |P_j| \]

\[ \leq \sum_{j=1}^{n-1} \frac{X_{jj}}{(n-1)X_{jj}} \cdot \frac{1}{2} \cdot \min_{(i,j)} \delta_{ij} \]

\[ = \frac{n-1}{(n-1)} \cdot \frac{1}{2} \cdot \min_{(i,j)} \delta_{ij} \]

\[ = \frac{1}{2} \min_{(i,j)} \delta_{ij} \]

Hence the line flow constraints will be satisfied.
Conclusion: we obtain a steady-state security region
\[ \mathcal{R} = \{ P : |P_j| \leq \frac{1}{(n-1)X_{jj}} \cdot \frac{1}{2} \cdot \text{Min} \delta_{ij} \} \text{ for } j = 1, 2, \ldots, n - 1 \]

Question: why \( X_{ij} \geq 0 \) and \( X_{ii} \geq X_{ij} \) (or \( X_{jj} \geq X_{ij} \))? The node admittance matrix is \( B \). So we can build an analog based on the nodal analysis.

\[ \begin{array}{c}
  i \\
  \hline \\
  B_{ij} \\
  \hline \\
  j \\
\end{array} \]

\[ \begin{array}{c}
  N \\
\end{array} \]

\( B\theta = P \) \( \theta \rightarrow \) nodal voltage
\( P \rightarrow \) current injections

Then \( \theta = XP \)
Suppose \( P^i = [0 \ldots 0 1 0 \ldots 0]^T \) and \( \theta^i = XP^i \); then
Physically, $\theta$ can be interpreted as follows:

Node Voltages $X_{ii}$ $X_{ij}$

Now it is seen that all node voltages must be positive. Furthermore, the voltages at node $j$ ($j \neq i$) should satisfy $X_{ii} \geq X_{ij}$ ($j \neq i$).
We now show that $X_{ij} \geq X_{ij}$. Suppose there is (at least) one node $j$ with a node voltage $X_{ij} \geq X_{ii}$.

Take the node with the highest voltage $X_{ij}$. Obviously all currents leaving that node $j$ must have positive value. By KCL constraint, all currents sum to zero. Therefore all currents must be zero and all neighboring nodes must also have the highest node voltage. One can repeat the arguments for these neighboring nodes and find that if the network is connected, then all nodes must have the highest node voltage, which is of course, impossible. Hence we conclude that $X_{ii} \geq X_{ij}$ ($j \neq 1$).

The fact that $X_{ij} \geq 0$ can also be established by similar arguments.
We have now obtained a security region given by

\[ \mathcal{R}_0 : \frac{1}{(n-1)X_{jj}} \cdot \frac{1}{2} \cdot \min_{(i,j)} \delta_{ij} \leq P_j \leq \frac{1}{(n-1)X_{jj}} \cdot \frac{1}{2} \cdot \min_{(i,j)} \delta_{ij} \]

\[ (j=1,2,\ldots, n-1) \]

It is noted that when \( n \) is large or there is a line with a very low capacity limit \( \delta_{ij} \), then the region would be quite small – conservative estimate of the security region. To resolve this problem, one can try to expand the security region by adding and incrementing \( \varepsilon \) to each dimension of the region

\[ \mathcal{R}_\varepsilon : -P_j^m - \varepsilon \leq P_j \leq P_j^M + \varepsilon \quad j=1,2,\ldots, n-1 \]
Construction of Maximal HYPERBOX

1. Base point
2. Initial region
3. Expansion
4. Maximality
At this point, one has to ensure that \( \mathcal{R}_\varepsilon \) is indeed a security region, i.e., all line flow constraints must be satisfied. To show this, one can check if all “corners” or \( \mathcal{R}_\varepsilon \) lie in \( \mathcal{R}_p \) (convexity). For an \((n-1)\)-dimensional hyperbox, this means we have to check the line flow constraints for \(2^{n-1}\) vectors of injections. Practically, this is infeasible when \(n\) is not small. In order to reduce the computation, the following observation is useful.
FACT: Suppose $\mathcal{R}_\varepsilon = \{ P : P^m \leq P \leq P^M \}$ then $\mathcal{R}_\varepsilon \subset \mathcal{R}_P$

IF AND ONLY IF for EVERY line $k$ connecting buses $i$ and $j$, the following inequality hold:
\[
\sum_{l=1}^{n-1} (X_{il} - X_{jl}) P^k_l \leq \delta_k \quad (*)
\]
\[
\sum_{l=1}^{n-1} (X_{il} - X_{jl}) P^k_l \geq -\delta_k \quad (+)
\]

where
\[
P^k_l = \begin{cases} 
P^M_l & \text{if } X_{il} - X_{jl} \geq 0 \\
P^m_l & \text{if } X_{il} - X_{jl} < 0
\end{cases}
\]
\[
P^k_l = \begin{cases} 
P^M_l & \text{if } X_{il} - X_{jl} < 0 \\
P^m_l & \text{if } X_{il} - X_{jl} \geq 0
\end{cases}
\]

Proof: Consider the DC load flow
\[
\theta_i = \sum_{l=1}^{n-1} X_{il} P_l
\]
If line $k$ connects buses $i$ and $j$, then

$$\theta_i - \theta_j = \sum_{l=1}^{n-1} (X_{il} - X_{jl}) P_l$$

Suppose $\mathcal{R}_\varepsilon = \{P: P^m \leq P \leq P^M\}$. It can be seen that for every $P \in \mathcal{R}_\varepsilon$,

$$\sum_{l=1}^{n-1} (X_{il} - X_{jl}) \tilde{P}_l \leq \sum_{l=1}^{n-1} (X_{il} - X_{jl}) P_l \leq \sum_{l=1}^{n-1} (X_{il} - X_{jl}) \hat{P}_l$$

As a result, if equations (*) and (+) hold, then

$$-\delta_k \leq \sum_{l=1}^{n-1} (X_{il} - X_{jl}) P_l \leq \delta_k \quad \forall P \in \mathcal{R}_\varepsilon$$

or:

$$-\delta_k \leq \theta_i - \theta_j \leq \delta_k$$

If for every line $k$ equations (*) and (+) are satisfied, then all line flow constraints will be respected.

For every injection vector $P$ in $\mathcal{R}_\varepsilon$. Hence the region $\mathcal{R}_\varepsilon$ is a subset of $\mathcal{R}_P$. 
Initial Region

Proposition 1, Given $P^0$, if the following conditions are satisfied, then $P = P^0 + \Delta P \in R_p$.

\[
\begin{aligned}
\text{Max} \left\{ \frac{-\Phi}{(N-1)X_{ii}}, P^m_i - P^0_i, -\frac{P^M_N - P^0_N}{N-1}, -\frac{P^0_i - P^m_N}{N-1} \right\} \leq \Delta P_i \leq \\
\text{Min} \left\{ \frac{\Phi}{(N-1)X_{ii}}, P^M_i - P^0_i, \frac{P^M_N - P^0_N}{N-1}, \frac{P^0_i - P^m_N}{N-1} \right\} \\
i = 1, 2, ..., N - 1
\end{aligned}
\]

Where $\phi = \frac{1}{2} \text{Min} \left\{ \left| \delta^1_i \right|, \left| \delta^2_i \right| \right\}$ and $X = B^{-1}$

Proof:

\[
\begin{aligned}
\theta &= X P \\
-\delta_{ij} &\leq \theta_i - \theta_j \leq \delta_{ij}
\end{aligned}
\]
DC load flow assumes that transmission lines are lossless. Hence

\[ P_1 + P_2 + \ldots + P_n = 0 \]

Or

\[ P_n = -\sum_{i=1}^{n-1} P_i \]

Generation limits:

\[
\begin{cases}
   P_i^m \leq P_i^0 + \Delta P_i \leq P_i^M & i = 1, \ldots, n-1 \\
   \sum_{i=1}^{n-1} \Delta P_i \leq P_n^M - P_n^0 & i = n
\end{cases}
\]

Then

\[
-\left( P_i^0 - P_i^m \right) \leq P_i \leq P_i^M - P_i^0 & i = 1, \ldots, n-1 \\
|\Delta P_i| \leq \text{Min}\left\{ P_i^M - P_i^0, P_i^0 - P_i^m \right\} & i = 1, \ldots, n-1
\]
$$- \sum_{i=1}^{n-1} \Delta P_i \leq \sum_{i=1}^{n-1} |\Delta P_i|$$

$$P_n = \Delta P_n + P_n^0$$

$$\leq \sum_{i=1}^{n-1} \frac{P_n^M - P_n^0}{n - 1}$$

$$= P_n^M - P_n^0 + P_n^0$$

$$- \sum_{i=1}^{n-1} \Delta P_i \geq - \sum_{i=1}^{n-1} |\Delta P_i|$$

$$\leq - \sum_{i=1}^{n-1} \frac{P_n^0 - P_n^m}{n - 1}$$

$$= - P_n^0 + P_n^m + P_n^0$$
Necessary and Sufficient Condition For Security HyperBox

**Proposition 2.** $D_p \subseteq R_p$ If and Only If

(I) For every line $K$ connecting buses $I$ and $J$

\[
\sum_{l=1}^{N-1} (X_{li} - X_{jl}) \Delta P_l \leq \delta_k^1
\]

\[
\sum_{l=1}^{N-1} (X_{li} - X_{jl}) \Delta P_l \leq \delta_k^2
\]

(II) $\Delta \bar{P}_l^M \leq P_l^M - P_l^0 \quad l = N_l + 1, \ldots, N - 1$

\[
\Delta \bar{P}_l^m \geq P_l^m - P_l^0
\]

\[
- \sum_{l=1}^{N-1} \Delta \bar{P}_l^m \leq P_N^M - P_N^0
\]

\[
- \sum_{l=1}^{N-1} \Delta \bar{P}_l^M \geq P_N^m - P_N^0
\]

$P_l^M, P_l^m$: Generation limits
**Algorithm A: (Construction of Security Regions)**

Step 0: Compute the initial Region $D_p^0$

Set $K=0$

Step 1: Set $K=K+1$ and $V_I^M = 1$

Step 2: Compute

$\Delta P_{I,K}^M = \Delta P_{I,K-1}^M - 1 + \varepsilon V_I^M$

$\Delta P_{I,K}^m = \Delta P_{I,K-1}^m - 1 - \varepsilon V_I^m$

Define

$D^k_p = \left\{ \Delta P : \Delta P_{I,K}^m \leq \Delta P_I \leq \Delta P_{I,K}^M \right\}$

Step 3: If $D^k_p \subset R_p$, set $K=K+1$ and go to step 2. Else find the critical vertices which violate security constraints and reset the bounds and set the corresponding $V_I^m$ or $V_I^M$ to zero.

Step 4: If $V_I^m = V_I^M = 0$, then set $K=K+1$ and go to step 5. Else, set $K=K+1$ and go to step 2.
Step 5: If $\varepsilon < \sigma$, then stop. Else set $\varepsilon = \frac{1}{10} \varepsilon$ and go to step 1.
Algorithm B: Maximal Regions

Step 0: Set \( l = 0, \hat{D}_P = D_P \)

Step 1: Set \( l = l + 1 \)

Compute \( \mathcal{E}_l^M \) and \( \mathcal{E}_l^m \) by Max \( \mathcal{E}_l \)

Subject to \( D_P(\mathcal{E}_l) \subseteq R_P \)

Where \( D_P(\mathcal{E}_l) \) is

\[
\left\{ \Delta P : \Delta P_l^m \leq \Delta P_l \leq \Delta P_l^M + \mathcal{E}_l, \Delta P_K^m \leq \Delta P_K \leq \Delta P_K^M, K \neq l \right\}
\]

or

\[
\left\{ \Delta P : \Delta P_l^m - \mathcal{E}_l \leq \Delta P_l \leq \Delta P_l^M, \Delta P_K^m \leq \Delta P_K \leq \Delta P_K^M, K \neq l \right\}
\]
Step 2: Set \( P_l^M = P_l^M + \varepsilon_l^M \), \( P_l^m = P_l^m + \varepsilon_l^m \)

If \( l = N - 1 \), stop.

Else, go to step 1
Features

- Algorithm. A is very efficient
- Algorithm. A produces near maximal boxes.
- Computation of algorithm B (use algorithm A)
Size of Hyperboxes
A measure for the size of hyperboxes

\[
\frac{\text{Upper Bound} - \text{Lower Bound of } P_I}{\text{Base Value of } P_I} \times 100\%
\]
Numerical results:

1. 6-bus system

<table>
<thead>
<tr>
<th>Base Value $p^0$</th>
<th>Initial Region Line Limits Only</th>
<th>Max Security Region Line Limits Only</th>
<th>Max Security Region Line and Generation Only</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.4</td>
<td>-0.0282 $\leq \Delta P_1 \leq 0.0282$</td>
<td>-0.53385 $\leq \Delta P_1 \leq 0.42934$ (40%)</td>
<td>-2.600 $\leq P_1 \leq$ -2.100 (21%)</td>
</tr>
<tr>
<td>3.2</td>
<td>-0.0265 $\leq \Delta P_2 \leq 0.0265$</td>
<td>-0.53210 $\leq \Delta P_2 \leq 0.42758$ (30%)</td>
<td>2.900 $\leq P_2 \leq$ 3.600 (22%)</td>
</tr>
<tr>
<td>-1.6</td>
<td>-0.0216 $\leq \Delta P_3 \leq 0.0216$</td>
<td>-0.44345 $\leq \Delta P_3 \leq$ 1.86916 (145%)</td>
<td>-1.800 $\leq P_3 \leq$ -1.400 (25%)</td>
</tr>
<tr>
<td>-2.4</td>
<td>-0.0326 $\leq \Delta P_4 \leq 0.0326$</td>
<td>-0.43781 $\leq \Delta P_4 \leq$ 0.46245 (38%)</td>
<td>-2.600 $\leq P_4 \leq$ -2.100 (21%)</td>
</tr>
<tr>
<td>1.649</td>
<td>-0.0231 $\leq \Delta P_5 \leq 0.0231$</td>
<td>-0.52871 $\leq \Delta P_5 \leq$ 0.42420 (58%)</td>
<td>1.549 $\leq P_5 \leq$ 2.000 (27%)</td>
</tr>
</tbody>
</table>

Security Region Results of 6-Bus System
2. IEEE 30 Bus System

<table>
<thead>
<tr>
<th>Initial Region Line Limits Only</th>
<th>Max Security Region Line Limits Only</th>
<th>Max Security Region Line and Generation Only</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-0.00252 \leq \Delta P_1 \leq 0.00252)</td>
<td>(-0.14146 \leq \Delta P_1 \leq 0.92278) ((305%))</td>
<td>(0.210 \leq P_1 \leq 0.45) ((69%))</td>
</tr>
<tr>
<td>(-0.00079 \leq \Delta P_2 \leq 0.00079)</td>
<td>(-0.53210 \leq \Delta P_2 \leq 0.92105) ((163%))</td>
<td>(-0.649 \leq P_2 \leq 0.349) ((46%))</td>
</tr>
<tr>
<td>(-0.00093 \leq \Delta P_3 \leq 0.00093)</td>
<td>(-0.13986 \leq \Delta P_3 \leq 0.32119) ((77%))</td>
<td>(0.005 \leq P_3 \leq 0.012) ((117%))</td>
</tr>
<tr>
<td>(-0.00029 \leq \Delta P_4 \leq 0.00029)</td>
<td>(-0.13922 \leq \Delta P_4 \leq 0.32055) ((299%))</td>
<td>(0.100 \leq P_4 \leq 0.212) ((73%))</td>
</tr>
<tr>
<td>(-0.00033 \leq \Delta P_5 \leq 0.00033)</td>
<td>(-0.13925 \leq \Delta P_5 \leq 0.32058) ((156%))</td>
<td>(0.189 \leq P_5 \leq 0.55) ((122%))</td>
</tr>
</tbody>
</table>

Security Region of 30-Bus System (generations only)
### MAX SECURITY REGION

<table>
<thead>
<tr>
<th>BUS</th>
<th>BOUNDS</th>
<th>BUS</th>
<th>BOUNDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>-0.045 ≤ P6 ≤ 0</td>
<td>17</td>
<td>-0.052 ≤ P17 ≤ 0</td>
</tr>
<tr>
<td>7</td>
<td>-0.097 ≤ P7 ≤ 0</td>
<td>18</td>
<td>-0.102 ≤ P18 ≤ 0</td>
</tr>
<tr>
<td>9</td>
<td>-0.229 ≤ P9 ≤ -0.105</td>
<td>19</td>
<td>-0.029 ≤ P19 ≤ 0</td>
</tr>
<tr>
<td>11</td>
<td>-0.065 ≤ 11 ≤ 0</td>
<td>20</td>
<td>-0.182 ≤ P20 ≤ -0.064</td>
</tr>
<tr>
<td>12</td>
<td>-0.132 ≤ P12 ≤ -0.011</td>
<td>22</td>
<td>-0.052 ≤ P22 ≤ 0</td>
</tr>
<tr>
<td>13</td>
<td>-0.082 ≤ P13 ≤ 0</td>
<td>23</td>
<td>-0.094 ≤ P23 ≤ 0</td>
</tr>
<tr>
<td>14</td>
<td>-0.102 ≤ P14 ≤ 0</td>
<td>25</td>
<td>-0.041 ≤ P25 ≤ 0</td>
</tr>
<tr>
<td>15</td>
<td>-0.055 ≤ P15 ≤ 0</td>
<td>28</td>
<td>-0.024 ≤ P28 ≤ 0</td>
</tr>
<tr>
<td>16</td>
<td>-0.097 ≤ P16 ≤ 0</td>
<td>29</td>
<td>-0.106 ≤ P29 ≤ 0</td>
</tr>
</tbody>
</table>

Security Region of 30-Bus System (loads only)
APPLICATION

GIVEN LOAD DEMANDS

\[ D_P^I \cap D_P^J \neq \emptyset \]  \rightarrow  CAN MEET THE LOAD
APPLICATION
GIVEN GENERATION RESERVES

\[ P_I^m \leq P_I \leq P_I^M \]

LOAD UNCERTAINTIES

\[ \ell 1 - \varepsilon < \ell < \ell 1 + \varepsilon \]

LINE FLOW SECURITY

AVAILABLE GEN.

LOAD W/ UNCERT AINTY
CONCLUSION

- AN EFFICIENT ALGO. IS PROPOSED.
- SIZE OF HYPERBOXES IS REALISTIC.
- $D_p$ MAY NOT BE CLOSE TO $R_p$.
- THE ALGO. CAN BE EXTENDED TO LINEARIZED POWER FLOW WITH P AND Q.
Further Information