Example for Linear Programming and Nonlinear Programming
Fixed problem:
\[ f(x) = 0 \]

Optimization problem:
\[ \min f(x) \]

What our world is

How to make our world better
Optimal Power Flow

General form

\[
\begin{align*}
\text{Min } f & \quad (u, x, z) \\
\text{s.t. } & \quad g(u, x, z) = 0 \quad \text{(Power flow equation)} \\
& \quad h(u, x, z) \leq 0 \quad \text{(Operating limits)}
\end{align*}
\]

- \( u \): decision variables (independent variables)
  - Generator voltage magnitude & real power
  - Voltage magnitude & angle at slack bus
  - Real power flow through dc lines
  - Phase angles across phase-shifting transformers
  - Turn ratios of tap-changing transformers
  - Admittances of variable reactors and switched capacitor banks
  - breaker positions by which the network can be reconfigured
Optimal Power Flow

- $\mathbf{z}$: exogenous variables (specified)
  - Real & reactive demands at load buses
  - Tie line flows
  - Admittance matrix

- $\mathbf{x}$: dependent variables
  - Real & reactive power at slack bus
  - Reactive power & angle at PV buses
  - Voltage magnitudes & angles at PQ buses
Optimal Power Flow

General form

\[ \min f \left( u_\_, \ x_\_, \ z_\_ \right) \]

\[ \text{s.t.} \quad \begin{align*}
  g\left( u_\_, \ x_\_, \ z_\_ \right) &= 0 \quad (\text{Power flow equation}) \\
  h\left( u_\_, \ x_\_, \ z_\_ \right) &\leq 0 \quad (\text{Operating limits})
\end{align*} \]

How to form:

Cost function

Power flow equalities

Operating limit inequalities

Three ways to form the OPF problem

Linear

Or

Nonlinear
6-bus system
Linear Programming OPF (LPOPF)

Method 1

- **Objective Function** — Linear Cost Function
  \[ F_i(P_i) = s_i(P_i - P_i^{\text{min}}) + F_i(P_i^{\text{min}}) \Rightarrow \min \sum_i s_i P_i \]

- **Power Flow equality**: Power balance equation
  \[ P_1 + P_2 + P_3 = P_{\text{load}} + P_{\text{losses}} = 0 \text{ in DC power flow} \]

- **Operating constraints**:
  \[
  \begin{cases}
  P = B\theta \\
  -\delta \leq A^T \theta \leq \delta \\
  -\delta \leq A^T B^{-1} P \leq \delta \\
  \end{cases}
  \]
  \[ P^{\text{min}} \leq P \leq P^{\text{max}} \]
Linear Programming OPF (LPOPFP)

- 6-bus system:

\[-\delta \leq A^T B^{-1} P \leq \delta\]

\[\begin{bmatrix}
P_2 \\
P_3 \\
P_4 \\
P_5 \\
P_6 \\
\end{bmatrix}\]

Independent variables:

Exogenous variables (specified)

How to eliminate the exogenous variables?

\[\Delta P_k^i = G_{k-i} \Delta P_i\]

Linear sensitivity coefficient

\(G_{k-i}\): Generation shift distribution factor, GSDF
Linear Programming OPF (LPOPF)

- 6-bus system:

\[
\text{min} \quad 12.4685P_1 + 11.2887P_2 + 11.8333P_3
\]
\[
s.t. \quad P_1 + P_2 + P_3 = 210\text{MW}
\]
\[
50 \leq P_1 \leq 200
\]
\[
37.5 \leq P_2 \leq 150
\]
\[
45 \leq P_3 \leq 180
\]
\[
-\delta \leq A^T B^{-1} P \leq \delta
\]

```matlab
LineLimit = [0.3;0.5;0.4;0.2;0.4;0.2;0.3;0.2;0.6;0.2;0.2];

f = [12.4685;11.2887;11.8333]';

% line limit constraints
AA = Ain' * inv(B);
bb = inv(b) * LineLimit; % Line delta limits
AAA = [zeros(11,1),AA(:,1:2)];
deltab = AA(:,3:5)*[-0.7;-0.7;-0.7];
Aineq = [AAA:-AAA];
bineq = [bb-deltab;bb+deltab];

% power balance equality constraint
Aeq = [1;1;1]';
beq = 2.1;

% power generation limit constraints
lb = [50,37.5,45]/100;
ub = [200,150,180]/100;

x = linprog(f,Aineq,bineq,Aeq,beq,lb,ub)
```
**Result:**

\[
\begin{align*}
\text{min} & \quad 12.4685 P_1 + 11.2887 P_2 + 11.8333 P_3 \\
s.t. & \quad P_1 + P_2 + P_3 = 210 MW \\
& \quad 50 \leq P_1 \leq 200 \\
& \quad 37.5 \leq P_2 \leq 150 \\
& \quad 45 \leq P_3 \leq 180 \\
& \quad -\delta \leq A^T B^{-1} P \leq \delta
\end{align*}
\]

Without line flow constraints:

P1* = 50, P2* = 115, P3* = 45

With line flow constraints:

P1 * = 79.8, P2* = 83.01, P3* = 47.19
**Linear Programming OPF (LPOPF)**

**Method 2**

- **Objective Function** — **Piecewise Linearization**

\[
F_i(P_i) = F_i(P_i^\text{min}) + s_{i1}P_{i1} + s_{i2}P_{i2} + s_{i3}P_{i3}
\]

\[
0 \leq P_{ik} \leq P_{ik}^+, k = 1, 2, 3
\]

\[
P_i = P_i^\text{min} + P_{i1} + P_{i2} + P_{i3}
\]
**Linear Programming OPF (LPOPF)**

**TABLE 13.2  Generator Unit Break Point MWs**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Break Point 1 (unit min)</th>
<th>Break Point 2</th>
<th>Break Point 3</th>
<th>Break Point 4 (unit max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
<td>100</td>
<td>160</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>37.5</td>
<td>70</td>
<td>130</td>
<td>150</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>90</td>
<td>140</td>
<td>180</td>
</tr>
</tbody>
</table>

**TABLE 13.3  Generator Cost Curve Segment Slope**

<table>
<thead>
<tr>
<th>Generator</th>
<th>$s_{i1}$</th>
<th>$s_{i2}$</th>
<th>$s_{i3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.4685</td>
<td>13.0548</td>
<td>13.5875</td>
</tr>
<tr>
<td>2</td>
<td>11.2887</td>
<td>12.1110</td>
<td>12.8222</td>
</tr>
<tr>
<td>3</td>
<td>11.8333</td>
<td>12.5373</td>
<td>13.2042</td>
</tr>
</tbody>
</table>

**Objective Function**

$$[F_1(P_{1}^{\text{min}}) + 12.4685P_{11} + 13.0548P_{12} + 13.5878P_{13}] + [F_2(P_{2}^{\text{min}}) + 11.2887P_{21} + 12.1110P_{22} + 12.8222P_{23}] + [F_3(P_{3}^{\text{min}}) + 11.8333P_{31} + 12.5373P_{32} + 13.2042P_{33}]$$
Nonlinear Programming OPF (NLPOPF)

**Method 3**

- **Objective Function** — Quadratic cost function

\[ F_1(P_1) = 213.1 + 11.669P_1 + 0.00533P_1^2 \text{ R/h} \]
\[ \text{with limits of: } 50.0 \text{ MW} \leq P_1 \leq 200.0 \text{ MW} \]

\[ F_2(P_2) = 200.0 + 10.333P_2 + 0.00889P_2^2 \text{ R/h} \]
\[ \text{with limits of: } 37.5 \text{ MW} \leq P_2 \leq 150.0 \text{ MW} \]

\[ F_3(P_3) = 240.0 + 10.833P_3 + 0.00741P_3^2 \text{ R/h} \]
\[ \text{with limits of: } 45.0 \text{ MW} \leq P_3 \leq 180.0 \text{ MW} \]

**Objective function**

\[ H = \text{diag}([0.00533, 0.00889, 0.00741])/2*100; \]
\[ f = [11.669; 10.333; 10.833]'; \]
\[ x = \text{quadprog}(H,f,Aineq,bineq,Aeq,beq,lb,ub); \]

P1* = 76.86,  
P2* = 76.41,  
P3* = 56.73
Nonlinear Programming OPF (NLPOPF)

Method 4

- Constraints:
  - Can we consider network losses?
    \[ P_1 + P_2 + \cdots + P_n = P_{\text{load}} + P_{\text{losses}} \]
  - Can we consider N-1 security?
  - Can we consider bus voltage constraints?
  - ...

Further Information

Appendix: Kuhn-Tucker Condition

\[ \min f(x) \]
\[ s.t. g(x) = 0 \]
\[ h(x) \leq 0 \]

**Kuhn-Tucker Condition:**
Let \( \mathbf{x}^* \) be a relative minimum point.
Suppose \( \mathbf{x}^* \) is a regular point for the constraints. Then there exist \( \mathbf{\lambda} \) and \( \mathbf{\mu} \) such that

\[ \frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}^*) + \mathbf{\lambda}^T \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*) + \mathbf{\mu}^T \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*) = 0 \]
\[ \mathbf{\mu}^T h(\mathbf{x}^*) = 0 \]
\[ \mathbf{\mu} \geq 0 \]
Appendix: Kuhn-Tucker Condition

* Example:

\[
\begin{align*}
\min x_1^2 - 2x_1 + x_2^2 + 1 \\
\text{s.t. } h_1(x_1, x_2) &= x_1 + x_2 \leq 0 \\
h_2(x_1, x_2) &= x_1^2 - 4 \leq 0
\end{align*}
\]

* Kuhn-Tucker Condition:

1. \( \lambda_1 \geq 0, \lambda_2 \geq 0 \)
2. \( \lambda_1 h_1(x_1, x_2) = \lambda_1 (x_1 + x_2) = 0 \)
   \[ \lambda_2 h_2(x_1, x_2) = \lambda_2 (x_1^2 - 4) = 0 \]
3. \( 2x_1 - 2 + \lambda_1 + 2\lambda_2 x_1 = 0 \)
   \[ 2x_2 + \lambda_1 = 0 \]
Appendix: Kuhn-Tucker Condition

(1) \( \lambda_1 \geq 0, \lambda_2 \geq 0 \)

(2) \( \lambda_1 h_1(x_1, x_2) = \lambda_1 (x_1 + x_2) = 0 \)

\( \lambda_2 h_2(x_1, x_2) = \lambda_2 (x_1^2 - 4) = 0 \)

(3) \( 2x_1 - 2 + \lambda_1 + 2\lambda_2 x_1 = 0 \)

\( 2x_2 + \lambda_1 = 0 \)

* (1)(2)(3) only have 2 solutions:

\( x^*_1 = 1, x^*_2 = 0, \lambda^*_1 = 0, \lambda^*_2 = 0 \)

\( x^*_1 = \frac{1}{2}, x^*_2 = -\frac{1}{2}, \lambda^*_1 = 1, \lambda^*_2 = 0 \)

* The first solution must be discarded because \( x^* = (1, 0) \) is not feasible for the problem.

* On the other hand, the second solution is feasible and is the only solution of the optimal problem.