ON THE DYNAMICS OF
DIFFERENTIAL-ALGEBRAIC SYSTEMS SUCH AS
THE BALANCED LARGE ELECTRIC POWER SYSTEM

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Abstract

This paper summarizes some recent results on the state space structure in differential-algebraic systems of the form \( \dot{x} = f(x, y) \) and \( 0 = g(x, y) \) which are the slow dynamics of singularly perturbed nonlinear systems of the form \( \dot{x} = f(x, y) \) and \( \varepsilon \dot{y} = g(x, y) \). After restricting the analysis to the portion of the slow manifold with local stability in the fast variables, the boundary of the region of attraction is characterized under some Morse-Smale like assumptions. It is proved that these constrained singular systems do possess a nice dynamical structure both locally (in general) and globally (under nongeneric assumptions) and the structure is of direct practical significance. The dynamics of the balanced large power system is a primary example.

1. INTRODUCTION

The dynamics of the balanced large electric power system can be modeled by singularly perturbed differential equations of the form

\[
\Sigma_\varepsilon : \quad \dot{x} = f(x, y, p), \quad f : \mathbb{R}^{n+m+q} \rightarrow \mathbb{R}^n, \quad f \text{ is } C^\infty
\]

\[
\varepsilon \dot{y} = g(x, y, p), \quad g : \mathbb{R}^{n+m+q} \rightarrow \mathbb{R}^m, \quad g \text{ is } C^\infty
\]

with the lumped RLC representation of the transmission system [1]. A model of the form \( \Sigma_\varepsilon \) was earlier proposed in [2] for the power system under the assumption that the capacitors are absent in the system and was applicable to certain types of load models. Such limitations were eliminated in [1] using a notion of fast time varying phasors where dynamical equations of the form (1)–(2) were constructed for the balanced general large electric power system. It should be emphasized that singularly perturbed models of the form \( \Sigma_\varepsilon \) arise from a precise formulation of the lumped RLC network dynamics in [1], and it does not justify casually defined singular perturbations of the load flow equations which are commonly used in the literature.

Typical dynamic state variables \( x \) are the time dependent values of the generator voltages and rotor phases, the dynamic states of the control devices and dynamic load models, and the dynamic phasor states of the capacitor banks and large reactors. The fast dynamic variables \( y \) are typically the phasor states associated with the capacitors and inductors of the lumped RLC transmission
system. The fast state variables of the synchronous generator [2] and any fast dynamics of the
dynamic load models may also be modeled in the set of $y$ variables. The parameter space $P$
composed of system parameters (the system topography, i.e. what is energized, and equipment
constants e.g. inductances), and operating parameters (such as loads, generation, voltage set-points
etc.). System hard limits and saturation are not considered here.

In the precise formulation of (1)–(2), each capacitor and inductor in the lumped RLC represen-
tation of the transmission system define two dynamic states for describing their fast dynamics,
which however result in a very high dimensional system $\Sigma_e$ that is computationally not attractive.
Moreover since there exists a natural time-scale separation between the fast dynamic states of the
transmission system (in $y$) and the traditional quasi stationary phasor states (in $x$), the singular
perturbation theory can be used to simplify the analysis of $\Sigma_e$ [3].

Essentially the solutions of the system $\Sigma_e$ can be well-approximated by those of the correspond-
ing slow system $\Sigma_s$ defined by

$$
\Sigma_s : \quad \dot{x} = f(x, y, p), \quad f : \mathbb{R}^{n+m+q} \rightarrow \mathbb{R}^n, \quad f \text{ is } C^\infty 
$$

(3)

$$
0 = g(x, y, p), \quad g : \mathbb{R}^{n+m+q} \rightarrow \mathbb{R}^m, \quad g \text{ is } C^\infty
$$

(4)

provided certain stability requirements on the fast dynamics are satisfied [3]. Integral manifold
theory has also been applied extensively in the power system literature (e.g. [2]) previously for the
simplification of the power system models. Traditional power system dynamic analysis has been
based on the differential-algebraic systems of the form $\Sigma_s$ where roughly the network equations
$g = 0$ in (4) can be shown to reduce to the conventional load flow equations or the power balance
equations after some simplifications. But unlike the conventional quasi-stationary power system
models, the full model $\Sigma_e$ includes in it the fast dynamics as well, therefore the stability of the
slow system $\Sigma_s$ within the full dynamics, namely the component stability problem, can be directly
assessed from the overall system stability of $\Sigma_e$ [1]. Note that the lumped parameter RLC trans-
mission system representation is still limited in its scope and needs to be replaced by distributed
parameter - stray capacitor and inductance models for very fast phenomena.

Let us first define the constraint $g = 0$ in (4) (also known as the slow manifold) to be $L$,

$$
L := \{(x, y, p) \in \mathbb{R}^{n+m+q} : g(x, y, p) = 0\}
$$

(5)

From the singular perturbation theory [3], it follows that any trajectory of the system $\Sigma_e$ can be
well-approximated by that of the slow constrained system $\Sigma_s$ provided the “fast system” is locally
uniformly asymptotically stable along the trajectory, e.g. if the Jacobian $D_y g$ (i.e. $\frac{\partial g}{\partial y}$) has all its
eigenvalues in the open left half complex plane $\mathbb{C}^-$ along the trajectory, i.e.

1. if the trajectory stays away from the singular subset $S$ where the Jacobian $D_y g$ has zero
eigenvalues where $S$ is defined as

$$
S := \{(x, y, p) \in L : \Delta(x, y, p) := \det(D_y g)(x, y, p) = 0\}
$$

(6)
and

2. if the trajectory also stays away from the set $H$ where the Jacobian $D_yg$ has purely imaginary eigenvalues where $H$ is defined as

$$H := \{(x, y, p) \in L \setminus S : \sigma(D_yg)(x, y, p) \cap \mathcal{I} \neq \emptyset\}$$

(7)

where $\mathcal{I}$ denotes the imaginary axis in the complex plane.

The set $H$ named here the Hurwitzian surface and the singularity $S$ then outline the state space boundary where the component stability (or the stability of the fast dynamics) is lost.

The presence of the singular set (called the impasse surface in [6, 7]) in power system models has been noted before [4, 5, 6] and a detailed analysis of the constrained singular dynamics of the form $\Sigma_x$ has been presented elsewhere [8, 9]. When a trajectory reaches the singular set $S$, the solution of $\Sigma_x$ typically undergoes jumps, i.e., the trajectory moves very fast, almost instantaneously, to a different point on the constraint $L$. Such jump behavior has been studied in some classes of singularly perturbed systems [10, 11, 12]. However in general, the jump behavior can be extremely complex even in two dimensional ($n = 2$) systems near special singular points (canards, [13, 14]) and at the present time, it seems to us that there exists no general theory for analyzing the system behavior near the singular points $S$ for the large system $\Sigma_x$ (nonstandard singular perturbation, [14]).

The existence of the Hurwitzian set $H$ identified here can be readily confirmed mathematically in simplistic power system models but their existence in detailed power system models and their physical implications need careful analysis. Note that since the Jacobian $D_yg$ is typically not symmetric in power system models, the existence of the Hurwitzian set $H$ cannot be ruled out in general power system models. More on this later in Section 2.

Here it should be remembered that the model (1)–(2) itself is only an approximation for the large power system and is valid only under the assumptions that 1) the lumped parameter RLC representation of the transmission system is valid and 2) the three phase signals are balanced [1]. When the trajectories start moving very fast such as near the singular set and off the constraint $L$, these two assumptions become questionable. It can be shown that the inherent randomness in the reference phase of the modulated power system signals together with the breakdown of the assumptions imply that the power system dynamics becomes a true stochastic process near the singular set [1].

For the reasons mentioned above, the analysis in this paper will be restricted to studying the dynamics of the constrained system $\Sigma_x$, specifically, around the subset $L^s$ of the slow manifold $L$ which is component stable,

$$L^s := \{(x, y, p) \in L : \sigma(D_yg)(x, y, p) \subset \mathbb{C}^-\}$$

(8)
Here \( \sigma(D_yg) \) denotes the eigenspectrum of the Jacobian \( D_yg \). The component stability condition in (8) implies the local asymptotic stability of the fast dynamics in \( y \) near the constraint surface \( L^* \) for the singularly perturbed system \( \Sigma_e \). Any trajectory which lies entirely within \( L^* \) is indeed an excellent approximation for the singularly perturbed trajectory of \( \Sigma_e \) [3]. The dynamics of differential-algebraic systems of the form \( \Sigma_e \) has been analyzed extensively in [8, 9] and a Lyapunov theory for such constrained systems has been proposed in [6, 7]. The results in [8, 9] assumed the component stability in the state space since an explicit condition was unavailable for checking this assumption. With the development of such a condition in [1] (e.g. \( \sigma(D_yg) \subset \mathbb{C}^- \) being a sufficient condition), it becomes possible here to directly extend the results in [8, 9] now including this additional stability criterion. The extended results are presented in this paper.

The basic state space structure of the constrained system \( \Sigma_e \) is briefly summarized in Section 2, with the precise definitions to follow in Section 3. Stability regions are defined for the large power system in Section 4 and the boundary of the region of attraction is characterized in Section 4 under certain transversality conditions.

2. THE STRUCTURE OF THE STATE SPACE

The state space for the constrained system \( \Sigma_e \) displays a hierarchical structure summarized in Table 1. The constraint surface \( L \) is divided into connected components by the singular set \( S \) [8]. The state space components (called causal regions in [7]) are further decomposed into component stable and unstable regions by the interior component stability boundaries \( H \) where the Jacobian \( D_yg \) has purely imaginary eigenvalues. Simulation of simple power system models indicates that this set \( H \) is typically nonempty. The Hurwitzian surface \( H \) together with the singular surface \( S \) then divide the constraint \( L \) into connected regions where the fast dynamics is stable (component stable regions) and those where the fast dynamics is unstable (component unstable regions).

To be precise, the connected components of the set \( L^* \) are defined here the \textit{component stable regions} and the connected components of the complement \((L \setminus L^*) \setminus (S \cup H)\) are defined the \textit{component unstable regions}. The system operating point being a stable equilibrium point for \( \Sigma_e \) must belong to a component stable region. For the power system, the implications of the component unstable regions need a careful scrutiny. Even though the RLC dynamics which essentially defines the fast \( y \) dynamics in (2) should be an internally asymptotically stable system being a passive RLC network, since the coupling equations with the loads at the network terminals are nonlinear power balance equations in the power system, the dynamic equations (2) are nonlinear and hence it seems plausible that the network dynamics (2) could be locally unstable near some points of the network solutions depending on the load behavior. Actual load measurements on real large power systems [15, 16, 17] yield load models which indicate the presence of the singularity \( S \) hence these load models also indicate the possibility of the component unstable regions in the real power system. The possible presence of such component unstable regions was conjectured in [18]. Recent results in [1] provide their existence theoretically though their practical implications need further study.
Since the Jacobian $D_yg$ of the fast dynamics for the system $\Sigma_c$ has zero eigenvalues in the set $S$, the singular set $S$ can also be viewed as a static bifurcation for the fast dynamics, typically resulting in a saddle node type fold bifurcation [19] of the slow manifold solutions. In other words, two solutions of the constraint set $L$ typically meet and disappear at the singularity $S$ [12]. Similarly the Hurwitzian set $H$ may lead to a Hopf bifurcation [19] in the fast dynamics of $\Sigma_c$ provided certain transversality conditions are satisfied by the fast dynamics at a given Hurwitzian point in $H$. The Hopf bifurcation can in turn be either supercritical resulting in fast stable oscillations in the fast dynamics near the component unstable regions in the vicinity of the component stability boundary $H \cap \partial L^s$, or be subcritical corresponding to fast unstable oscillatory behavior away from the component unstable regions. Such bifurcation theoretic analysis of the interaction between the fast and slow dynamics near the sets $S$ and $H$ could be combined with the analysis of the slow dynamics $\Sigma_s$ near these points (to be presented in Section 3) for analyzing the overall jump phenomena in the large singularly perturbed system $\Sigma_c$. Moreover for the power system, the validity of the models of the form $\Sigma_c$ for global analysis needs to be re-evaluated [1], and the effect of the dynamic load models in their physical implications should be clarified.

As mentioned earlier, it can be shown that the slow dynamics $\Sigma_s$ indeed is an excellent approximation for the general power system analysis near the component stable region $L^s$. Therefore, in this paper, the analysis is restricted to the dynamics of the constrained system $\Sigma_s$ within component stable regions and the system behavior will be treated as unpredictable near the component unstable regions [1]. Component stable regions in turn contain regions of attractions of the slow dynamics $\Sigma_s$ and other regions where trajectories either converge to more complex limit points or may simply diverge.

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<th>Table 1. Structure of the state space</th>
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<td>Region</td>
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<td><strong>Boundary of the Region of attraction</strong></td>
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3. STATE SPACE DEFINITIONS

The dynamics of the constrained system $\Sigma_s$ will be analyzed in this section. A fixed parameter $p = p_0$ is considered for the state space analysis and the parameter $p$ will not be shown in the subsequent equations for simplicity. Also the state variables $(x, y)$ will be denoted together as $z$. The following sets essentially define the dynamical hierarchy of the state space:

\[
L = \{ (x, y) \in \mathbb{R}^{n+m} : g(x, y) = 0 \} \\
L^s = \{ (x, y) \in L : \sigma(D_y g)(x, y) \subset \mathbb{C}^- \} \\
M = \{ (x, y) \in L : \text{rank}(D_x g(x, y), D_y g(x, y)) = m \} \\
S = \{ (x, y) \in L : \Delta(x, y) := \det D_y g(x, y) = 0 \} \\
H = \{ (x, y) \in L \setminus S : \sigma(D_y g)(x, y) \cap \mathbb{I} \neq \emptyset \}
\]

$L$ is the constraint surface determined by the algebraic constraint. The slow system or the reduced system $\Sigma_s$ must stay on $L$ and henceforth all topological definitions are relative to $L$. $M$ is a regular subset of $L$ which, by the implicit function theorem, is an $n$-dimensional manifold. $L^s$ is the subset of the constraint surface $L$ where the fast dynamics is locally asymptotically stable. Therefore secure system operation is restricted to the subset $L^s$ of the constraint $L$. $L^s$ is bounded by the singular set $S$ (also called the impasse surface [7]) and the Hurwitzian surface $H$ within $L$. The singular surface $S$ can be characterized as the zero set of the determinant $\Delta(x, y)$ by definition (12) within $L$. Similarly it can be shown that a certain Hurwitzian determinant $\chi(x, y)$ defined as

\[
\chi(x, y) := \det(H_{n-1}(D_y g))(x, y)
\]

vanishes on the Hurwitzian surface $H \cap \partial L^s$ and here $H_{n-1}(A)$ denotes the classical Hurwitz matrix of order $n - 1$ for the characteristic polynomial of the matrix $A$ [20].

Lemma 1

\[
H \cap \partial L^s = \{ (x, y) \in \partial L^s \setminus S : \chi(x, y) = 0 \}
\]

Proof: In the boundary $H \cap \partial L^s$, the Jacobian $D_y g$ has purely imaginary eigenvalues with no zero eigenvalues, and all the remaining eigenvalues have negative real parts by the definition of $H$. Therefore the result follows by the Hurwitz determinant property from the classical system theory [20].

Q.E.D.

Therefore the component stability boundary $\partial L^s$ is entirely characterized by either $\Delta(x, y) = 0$ (the singular set $S$) or $\chi(x, y) = 0$ (the Hurwitzian set $H$). The set $S$ is especially significant since the system dynamics defined by $\Sigma_s$ is singular in $S$.

3.1 The singular set $S$ [8, 9]
Define the induced vector field of the constrained system $\Sigma_s$ on the constraint surface $L$ by

$$Z(x, y) := \begin{pmatrix} f(x, y) \\ -(D_y g(x, y))^{-1} D_x g(x, y) f(x, y) \end{pmatrix}$$

(16)

Clearly the vector field $Z$ is singular at the singularity $S$. The set $S$ can now be divided into the following three subsets which can be shown to have locally different solution behavior [8, 9]:

$$\Psi = \{(x, y) \in S : \kappa(x, y) := \text{adj}(D_y g(x, y)) D_x g(x, y) f(x, y) = 0\}$$

(17)

$$\Xi = \{(x, y) \in S \setminus \Psi : D_y \Delta(x, y) \kappa(x, y) = 0\}$$

(18)

$$R = S \setminus (\Psi \cup \Xi)$$

(19)

In the definition of $\kappa$, $\text{adj}(D_y g)$ denotes the classical matrix adjoint of the Jacobian $D_y g$. The subset $\Psi$ of $S$ is called the pseudo equilibrium surface. This high dimensional subset displays the behavior of an equilibrium surface for the vector field $Z$ (equation (16)) induced by the system $\Sigma_s$ and indeed becomes an equilibrium surface for an “equivalent” smooth dynamical system defined by multiplying the vector field $Z$ pointwise by the determinant $\Delta(x, y)$. Such a transformation was proposed in [10] for analyzing the local dynamics of the singular vector field $Z$. Using the classical matrix adjoint property, the resulting transformed vector field can be defined [8, 9] as

$$Z^T(x, y) := \begin{pmatrix} f(x, y) \Delta(x, y) \\ \text{adj}(D_y g)(x, y) D_x g(x, y) f(x, y) \end{pmatrix} = \begin{pmatrix} f(x, y) \Delta(x, y) \\ -\kappa(x, y) \end{pmatrix}$$

(20)

Since the transformation from $Z$ to $Z^T$ amounts to a pointwise rescaling of the vector field $Z$ by the factor $\Delta(x, y)$, it follows that the transformation is just a singular time rescaling along the trajectories. Therefore, the orbits or the trajectories of the vector field $Z$ (those of $\Sigma_s$), as integral curves, are identical to those of the those of the transformed vector field $Z^T$, excepting for a reversal of the orientation when $\Delta(x, y) < 0$. Hence the topological properties of the singular dynamics $Z$ such as the features of the stability boundary can be proved in the transformed smooth system for $Z^T$, and the results can be directly interpreted to the original singular system $\Sigma_s$. More details on the singular transformation and its properties can be seen in [8, 9].

From the definition of $\Psi$ and $Z^T$, it follows that the pseudo equilibrium surface $\Psi$ (17) is a true equilibrium surface for the transformed vector field $Z^T$. It is useful to distinguish the following subsets of $\Psi$:

$$N_\Psi = \{(x, y) \in \Psi \cap M : \text{rank}(D_y g) = m - 1 \text{ and the Jacobian of } Z^T \text{ has exactly two eigenvalues with nonzero real parts}\}$$

(21)

$$B_\Psi = \Psi \setminus N_\Psi$$

(22)

Typically, the Jacobian of $Z^T$ (restricted to $M$) has at least $(n - 2)$ zero eigenvalues at a point in $\Psi$. Hence $N_\Psi$ corresponds to the subset of $\Psi$ where the most regular type behavior takes place. It can be shown that near every point in $N_\Psi$ the integral curves of the system locally behave like
for a linear system, and the points in the set \( N_\psi \) will be called the \textit{nice} pseudo equilibrium points. The local dynamics near the points in the set \( B_\psi \) is more complex, and we call these points the \textit{bad} pseudo equilibrium points. Eventually certain assumptions will be imposed to make sure that the complement \( B_\psi \) is a truly lower dimensional subset.

The set \( \Xi \) corresponds to points where \( Z^T \) is tangent to the singularity. Henceforth points in \( \Xi \) will be called \textit{semi-singular}. Let

\[
N_\xi = \{ (x, y) \in \Xi : \text{rank} \begin{pmatrix} D_x g & D_y g \\ D_x \Delta & D_y \Delta \end{pmatrix} = m + 1, \\
\text{and } D_y \{(D_y \Delta) \kappa \kappa \neq 0\} \}
\]

\[
B_\xi = \Xi \setminus N_\xi
\]

The points in the set \( N_\xi \) are called the \textit{nice} semi-singular points, and the points in the set \( B_\xi \) the \textit{bad} semi-singular points. The condition on the second derivative in the definition of \( N_\xi \) ensures that trajectories do not cross over as they touch the singularity.

The set \( R \) is the complement of \( \Psi \) and \( \Xi \) in \( S \) and it consists of the singular points where exactly two trajectories of \( Z \) originate or converge transversal to \( R \) at infinite speeds. The points in \( R \) are called the \textit{transverse singular points}. By the definition of \( R \), at every point in \( R \), the term \((D_y \Delta) \kappa \) is nonzero. The set of transverse singular points where two trajectories are converging corresponds to the case when \((D_y \Delta) \kappa \) is negative, and the latter when \((D_y \Delta) \kappa \) is positive \cite{21}. Therefore the set of transverse singular points \( R \) can be divided into transverse \textit{sinks} \( R_{si} \) \((D_y \Delta) \kappa < 0\) and \textit{transverse sources} \( R_{so} \) \((D_y \Delta) \kappa > 0\) respectively.

\subsection{3.2 The Hurwitzian set \( H \)}

The analysis in this section is restricted to the component stability boundary \( H \cap \partial L^s \). Similar results follow for the other segments of \( H \). The induced vector field \( Z \) is smooth on the Hurwitzian surface \( H \) from equation (7). Let us divide the set \( H \) into two parts \( \mathcal{F} \) where the trajectories are transversal to \( H \) and \( \Upsilon \) where \( Z^T \) is tangential to \( H \):

\[
\mathcal{F} := \{(x, y) \in H \cap \partial L^s : v(x, y) := (D_x \chi, D_y \chi)(x, y)Z^T(x, y) \neq 0\}
\]

\[
\Upsilon := (H \cap \partial L^s) \setminus \mathcal{F}
\]

The points in the set \( \mathcal{F} \) are called the \textit{transversal Hurwitzian points} and their properties are summarized next:

\textbf{Lemma 2} If non-empty, then \( \mathcal{F} \) is an \((n - 1)\)-dimensional embedded submanifold of \( M \) and \( Z^T \) is transversal to \( \mathcal{F} \).

\textbf{Proof:} At every point in \( \mathcal{F} \), the local neighborhood of \( \mathcal{F} \) in \( L \) can be described by \( g(x, y) = 0 \) and \( \chi(x, y) = 0 \) with additional inequality conditions imposed. Since the constraint set \( L \) is invariant
under the transformed vector field $Z^T$, it follows that $(D_xg, D_yg)Z^T = 0$. Therefore

$$
\begin{pmatrix}
D_xg & D_yg \\
D_x\chi & D_y\chi
\end{pmatrix}
\begin{pmatrix}
0 \\
v(x, y)
\end{pmatrix}
= 0
$$

(27)

Let $z_0 \in \mathcal{X}$. As $z_0 \notin S$ (by the definition of $H$), it follows that $z_0 \in M$ (Lemma 2.3, [9]), therefore rank $(D_xg, D_yg) = m$. Also from the definition of $\mathcal{X}$, $v(z_0) \neq 0$ and so $(D_x\chi, D_y\chi)$ does not lie in the row span of $(D_xg, D_yg)$ at $z = z_0$. But then the Jacobian

$$
\begin{pmatrix}
D_xg & D_yg \\
D_x\chi & D_y\chi
\end{pmatrix}
$$

(28)

has maximal rank $(= m + 1)$ everywhere on $\mathcal{X}$. By the implicit function theorem and Lemma 1, the set $\mathcal{X}$ is therefore an $(n - 1)$-dimensional manifold. Equation (27) also shows that $Z^T$ is transversal to $\mathcal{X}$.

Q.E.D.

Moreover since any point in the Hurwitzian surface cannot be singular, then Lemma 2 proves the following: Given any transversal Hurwitzian point $z_0 \in \mathcal{X}$, there exists a single smooth trajectory of $Z$ which passes through $z_0$ and this trajectory is transversal to $\mathcal{X}$ at $z_0$. Note that the term $v(z_0)$ in equation (27) corresponds to the first time derivative of the function $\chi(x, y)$ at $z = z_0$ evaluated along this trajectory.

From the equation (27), it can be seen that the vector field $Z^T$ is tangential to the set $\Xi$. The points in the set $\Xi$ are called the semi-Hurwitzian points and the set $\Xi$ can be further divided into a nice subset $N_v$ which displays the most regular structure in $\Xi$ and the reminder $B_v$.

$$
N_v := \{(x, y) \in \Xi : (D_xv, D_yv)Z^T \neq 0\}
$$

(29)

$$
B_v := \Xi \setminus N_v
$$

(30)

The points in the set $N_v$ are called the nice semi-Hurwitzian points and those in $B_v$ the bad semi-Hurwitzian points. The properties of the set $N_v$ are summarized next:

**Lemma 3** If non-empty, $N_v$ is an $(n - 2)$-dimensional embedded submanifold of $M$ and $Z^T$ has a nonzero normal component at every point in $N_v$.

**Proof:** Note that for any point in $N_v$, the local neighborhood of $N_v$ in $L$ can be characterized by the equalities $g(x, y) = 0$, $\chi(x, y) = 0$ and $v(x, y) = 0$ along with some additional inequality conditions. Therefore the result can again be proved by the implicit function theorem and the proof is an extension of the argument in Lemma 2. Essentially it can be shown that in $N_v$,

$$
\begin{pmatrix}
D_xg & D_yg \\
D_x\chi & D_y\chi \\
D_xv & D_yv
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
s
\end{pmatrix}
$$

(31)
where \( \zeta = (D_x v, D_y v)^T \neq 0 \) in \( N_u \) by definition. Therefore it follows that the Jacobian

\[
J_u := \begin{pmatrix}
D_x g & D_y g \\
D_x \chi & D_y \chi \\
D_x v & D_y v
\end{pmatrix}
\]

is of rank \((m + 2)\) proving that the set \( N_u \) is of dimension \((n - 2)\) and that \( J_u \) is a normal vector field for \( N_u \). Again by \( (31) \), then the normal component of \( Z^T \) is nonzero. Q.E.D.

Therefore given any \( z_0 \in N_u \), there exists a single smooth trajectory of \( Z \) which passes through \( z_0 \) and is tangential to \( N_u \). Moreover this trajectory upon reaching \( N_u \) must immediately leave \( N_u \) by Lemma 3 and the second time derivative of \( \chi(x, y) \) at \( z = z_0 \) evaluated along this trajectory is given by \( \zeta \) in the equation \( (31) \). Since \( \zeta \neq 0 \) at \( z_0 \), then the sign of the function \( \chi(x, y) \) along the trajectory does not change at \( z_0 \) even though \( \chi(x, y) = 0 \) at \( z = z_0 \).

As introduced before, the constraint set \( L \) can be divided into connected regions by the singular surface \( S \) and the Hurwitzian surface \( H \). The connected components of the component stable set \( L^s \backslash (S \cup H) \) are called the component stable regions and the connected components of \( (L \backslash L^s) \backslash (S \cup H) \) are called the component unstable regions. The previous Lemma 3 proves that given any \( z_0 \in N_u \), the unique trajectory of \( Z^T \) which passes through \( z_0 \) tangential to \( N_u \) does so by staying in the same component stable or component unstable region because the function \( \chi(x, y) \) does not change sign. In section 4, under certain transversality conditions, it will be shown that these points anchor the stability boundary of the region of attraction along with the singular points, pseudo equilibrium points, semi-singular points, unstable equilibrium points and unstable periodic orbits as shown in Table 1.

### 3.3 Singular dynamics hierarchy theorem

The properties of the various subdivisions of the state space introduced in the previous two sections are now summarized as a theorem. The proof of the theorem for the singularity aspects can be seen in [8, 9] and the results for the Hurwitzian set have been already proved in Section 3.2 in the form of Lemmas 2 and 3.

**Theorem 1** *(Singular dynamics hierarchy theorem)* If the subsets are not empty, then

1. \( L \backslash S \) is an \( n \)-manifold. The solutions of the system \( \Sigma \) are the integral curves of the vector field \( Z \) on \( L \backslash S \).

2. \( R \) is an \((n-1)\)-dimensional manifold. At each point in \( R \), two trajectories approach or diverge transversally at infinite speeds.

3. \( N_\Psi \) is an \((n-2)\)-dimensional manifold. Ignoring orientations, the solution curves are locally equivalent to those of a linear system near the origin.
4. $N_z$ is an $(n-2)$-dimensional manifold. Trajectories are tangent to the singularity at $N_z$, but do not cross over.

5. $\mathcal{X}$ is an $(n-1)$-dimensional manifold. At each point in $\mathcal{X}$, a unique smooth trajectory passes through transversally.

6. $N_v$ is an $(n-2)$-dimensional manifold. Trajectories are tangential to the Hurwitzian set at $N_v$, but do not cross over.

\[ \square \]

### 4. STABILITY REGIONS

In this section, the boundary of region of attraction of a stable equilibrium point will be characterized under some Morse-Smale like assumptions. For the system $\Sigma_s$, equilibrium points and periodic orbits are defined as usual, with the proviso that they do not intersect the singular set $S$ or the Hurwitzian set $H$. When an equilibrium of $\Sigma_s$ is at the singularity, the resulting bifurcation has been analyzed in [8, 9, 22] as the singularity induced bifurcation for the constrained system $\Sigma_s$. Similarly when an equilibrium of the system $\Sigma_s$ is at the Hurwitzian surface $H$, it can be shown that the local stability at the equilibrium will undergo a change generically and the analysis of the resulting bifurcation called the Hurwitzian induced bifurcation will be presented elsewhere. Stable equilibrium points define the system operating points for general physical systems of the form $\Sigma_s$ such as the large power system. Since the definition of equilibrium points here is restricted to $L^s$, it is indeed sufficient to check the local stability in the constrained system $\Sigma_s$ for guaranteeing the local stability in the full system $\Sigma_c$.

Generally for systems of the form $\Sigma_c$, given a stable equilibrium point $z_s$, the set of all trajectories which converge to the point $z_s$ should be defined as the region of attraction for $z_s$. However in this paper, the definition of the region of attraction is restricted to those trajectories which converge to $z_s$ and stay entirely within the component stable region $C^s$ (i.e. the connected component of $L^s$) which contains $z_s$. This definition is motivated by the fact that for the power system, the system dynamics becomes unpredictable away from $L^s$ [1]. The region of attraction of a stable equilibrium $z_s$ is defined as

$$A = \{ z \in C^s : \Phi_t(z) \in C^s \forall t \geq 0, \Phi_t(z) \to z_s \text{ as } t \to \infty \} \quad (33)$$

where $\Phi_t(\cdot)$ denotes the flow of the induced vector field $Z$.

It should be pointed out that for general singularly perturbed systems of the form $\Sigma_c$, the definition of the region of attraction (33) only provides a conservative estimate of the full region of attraction. The possibility that a trajectory of the system $\Sigma_c$ reaching the stable equilibrium $z_s$ via a number of jumps [10, 11] from different component regions is ruled out in this definition. Further research is indicated on these questions for general systems of the form $\Sigma_c$. However for the power system, this definition seems to be the only practical definition because of fundamental
Similarly define the stable and unstable manifolds for equilibria and periodic orbits in the usual way, but with the restriction to the component stable region $C^s$ as in the case of the region of attraction. The stable and unstable manifolds for a pseudo equilibrium in $\Psi$, a semi-singular point in $\Xi$ and a semi-Hurwitzian point in $\Gamma$ is similar to those for the equilibrium, but the convergence may occur in finite time. As an example, the definitions for a nice semi-Hurwitzian point $z_0 \in \Gamma$ are shown below:

\begin{align}
W^s(z_0) &= \{ z \in C^s : \exists t_0 > 0 \text{ such that } \Phi_t(z) \in C^s \text{ for } 0 \leq t < t_0 \text{ and } \Phi_t(z) \to z_0 \text{ as } t \to t_0 \} \\
W^u(z_0) &= \{ z \in C^s : \exists t_0 < 0 \text{ such that } \Phi_t(z) \in C^s \text{ for } 0 \geq t > t_0 \text{ and } \Phi_t(z) \to z_0 \text{ as } t \to t_0 \}. \tag{34}
\end{align}

Note that ignoring the speed, the trajectories of the vector fields $Z$ and $Z^T$ are identical (topologically equivalent) in $C^s$ (Lemma 2.1, [9]). Since the trajectories (integral curves to be more precise) of $Z$ coincide with the trajectories of $Z^T$ within the component stable region $C^s$, the stability regions defined above for the flow $\Phi_t$ of $Z$ can be equivalently redefined in terms of the flow $\Phi_t^T$ of the transformed vector field $Z^T$, by restricting the trajectories of $Z^T$ to the component stable region $C^s$ as shown below for the region of attraction $A$.

\begin{equation}
A = \{ z \in C^s : \Phi_t^T(z) \in C^s \forall t \geq 0, \Phi_t^T(z) \to z_0 \text{ as } t \to \infty \} \tag{36}
\end{equation}

Hence the boundary of the region of attraction $\partial A$ (with respect to $L$) can be characterized equivalently for the transformed vector field $Z^T$.

By analyzing the stable manifolds as defined here for the transformed system $Z^T$, it can be shown that these definitions indeed result in smooth manifolds of trajectories for the points such as the nice pseudo equilibrium points $N_\psi$, nice semi-singular points $N_\xi$ and the nice semi-Hurwitzian points $N_\nu$. The detailed analysis of the local dynamics near $N_\psi$ and $N_\xi$ is available in [9, 22]. The semi-Hurwitzian points, a new mathematical concept introduced in this paper, can be treated similar to the semi-singular points considered in [8, 9, 22] and dynamical properties such as $\lambda$-lemmas can also be developed for the nice semi-Hurwitzian points. Here these details are omitted due to space limitations but can be seen in [23]. The main result of this paper, the characterization of the boundary $\partial A$, will be presented next after making some Morse-Smale like assumptions.

4.1 Assumptions

Assumptions (A): It will be assumed that the following assumptions are satisfied in the closure of the region of attraction $\overline{A}$:

(A0) The nice sets $N_\psi, N_\xi$ and $N_\nu$ are dense in $\Psi, \Xi$ and $\Gamma$, respectively. Furthermore, at every point $\psi \in \Psi, \xi \in \Xi$ or $z_0 \in \Gamma$, the sets $B_\psi, B_\xi$ and $B_\nu$ have dimension at most $(n - 3)$.
(A1) (a) Equilibria $z_0$ and periodic orbits $\gamma_0$ in the boundary of the region of attraction, $\partial A$, are hyperbolic.
(b) Except for sets of dimension at most $(n - 3)$, all pseudo saddles in $\partial A$ are transverse.

(A2) Stable manifolds from the set \{\(W^s(z_0), W^s(\gamma_0), W^s(N_C), W^s(B_C)\)\} intersect transversally with unstable manifolds from the set \{\(W^u(z_0), W^u(\gamma_0), W^u(N_C), W^u(B_C)\)\}. Here $N_C$ stands for a connected component of either $N_\psi$, $N_\xi$ or $N_\nu$ and $B_C$ stands for a connected component of $B_\psi$, $B_\xi$ or $B_\nu$.

(A3) All trajectories converge to an equilibrium point, a periodic orbit, a pseudo equilibrium point, a semi-singular point or a semi-Hurwitzian point.

It can be shown that generally assumptions (A0)–(A2) will be satisfied for the system $\Sigma_x$ [22, 9]. However Assumption (A3) is not generic, but will be satisfied if there exists a dissipating energy function as claimed next. The proof is a direct extension of the proof in [8, 9] which now includes the Hurwitzian component stability boundary as well.

**Lemma 4** If there exists a Lyapunov-function $V = V(x, y)$ such that

1. \((D_x f, D_y f)Z \leq 0\) in $A$;
2. $V$ is proper in the closure of $A$;
3. the $\omega$-limit set in $\overline{A}$ consists only of equilibria and periodic orbits;

then assumption (A3) is satisfied. All trajectories in $\partial A$ converge to equilibria, periodic orbits, pseudo equilibria, semi-singular points or semi-Hurwitzian points.

\[\square\]

4.2 Stability Boundary Theorem

The boundary of the region of attraction $\partial A$ is shown to consist of ten different segments when the Assumption (A) are satisfied.

**Theorem 2** (Stability Boundary Theorem) [23] Under assumptions (A) the stability boundary is composed of

1. Stable manifolds of unstable equilibria $z_0 \in \partial A$,

\[
\bigcup_{z_0 \in \partial A} \{z \in C^s : \Phi^T_t(z) \to z_0 \text{ as } t \to \infty \text{ and } \Phi^T_t(z) \in C^s \forall t > 0\}
\]  
(37)
2. Stable manifolds of unstable periodic orbits $\gamma_0 \subset \partial A$,
\[ \bigcup_{\gamma_0 \subset \partial A} \{ z \in C^s : \Phi^T_t(z) \to \gamma_0 \text{ as } t \to \infty \text{ and } \Phi^T_t(z) \in C^s \forall t > 0 \} \]  
(38)

3. Stable manifolds of transverse pseudo saddles $N_{tr.sa} \cap \partial A$,
\[ \bigcup_{\psi \in (N_{tr.sa} \cap \partial A)} \{ z \in C^s : \Phi^T_t(z) \to \psi \text{ as } t \to \infty \text{ and } \Phi^T_t(z) \in C^s \forall t > 0 \} \]  
(39)

4. Stable manifolds of semi-saddles $N_{se.sa} \cap \partial A$,
\[ \bigcup_{\xi \in (N_{se.sa} \cap \partial A)} \{ z \in C^s : \exists t_0 > 0 \text{ such that } \Phi^T_t(z) \to \xi \text{ as } t \to t_0 \text{ and } \Phi^T_t(z) \in C^s \forall 0 < t < t_0 \} \]  
(40)

5. Stable manifolds of other pseudo equilibria $(\Psi \setminus N_{se.sa}) \cap \partial A$,
\[ \bigcup_{\psi \in ((\Psi \setminus N_{tr.sa}) \cap \partial A)} \{ z \in \partial A : \Phi^T_t(z) \to \psi \text{ as } t \to \infty \text{ and } \Phi^T_t(z) \in C^s \forall t > 0 \} \]  
(41)

6. Stable manifolds of other semi-singular points $(\Xi \setminus N_{se.sa}) \cap \partial A$,
\[ \bigcup_{\xi \in ((\Xi \setminus N_{se.sa}) \cap \partial A)} \{ z \in \partial A : \exists t_0 > 0 \text{ such that } \Phi^T_t(z) \to \xi \text{ as } t \to t_0 \text{ and } \Phi^T_t(z) \in C^s \forall 0 < t < t_0 \} \]  
(42)

7. Transversal singular boundary pieces $R \cap \partial A$,

8. Stable manifolds of semi-Hurwitzian saddles $N_{se.Hu.sa} \cap \partial A$,
\[ \bigcup_{z_0 \in (N_{se.Hu.sa} \cap \partial A)} \{ z \in C^s : \exists t_0 > 0 \text{ such that } \Phi^T_t(z) \to z_0 \text{ as } t \to t_0 \text{ and } \Phi^T_t(z) \in C^s \forall 0 < t < t_0 \} \]  
(43)

9. Stable manifolds of other semi-Hurwitzian points $(\Upsilon \setminus N_{se.Hu.sa}) \cap \partial A$,
\[ \bigcup_{z_0 \in ((\Upsilon \setminus N_{se.Hu.sa}) \cap \partial A)} \{ z \in \partial A : \exists t_0 > 0 \text{ such that } \Phi^T_t(z) \to z_0 \text{ as } t \to t_0 \text{ and } \Phi^T_t(z) \in C^s \forall 0 < t < t_0 \} \]  
(44)

10. Transversal Hurwitzian boundary pieces $\Xi \cap \partial A$,
In other words,
\[ \partial A = \cup_{x_0} \theta A W^s(x_0) \cup y_0 \in \theta A W^s(y_0) \cup W^s(N_{tr,sa} \cap \partial A) \]
\[ \cup W^s(N_{sc,sa} \cap \partial A) \cup (W^s(\Psi \setminus N_{tr,sa}) \cap \partial A) \]
\[ \cup (W^s(\Xi \setminus N_{sc,sa}) \cap \partial A) \cup (R \cap \partial A) \cup (\Xi \cap \partial A) \]
\[ \cup W^s(N_{sc,Hu,sa} \cap \partial A) \cup (W^s(T \setminus N_{sc,Hu,sa}) \cap \partial A) \]  \hspace{1cm} (45) \]

The proof of the theorem is quite technical based on the standard \( \lambda \)-lemma type proofs and the invariance properties. The proofs connected with the singularity \( S \) and its subsets \( \Psi \) and \( \Xi \) can be seen in [9]. The additional lemmas which extend the result in [9] to the transversal Hurwitzian boundary and the semi-Hurwitzian boundary pieces are similar to the arguments for the transversal singular points and the semi-singular points respectively and can be seen in [23]. Necessary and sufficient conditions can be derived for checking the presence of the variety of anchor points of the stable manifolds similar to such results in [9] and the quasi-stability boundary \( \partial A \) can also be characterized under similar assumptions. Rigorous energy function estimates of the region of attraction can be developed from Theorem 2 similar to the results in [9].

Even though the equations in Theorem 2 seem complicated, the structure of the stability boundary is essentially simple as shown in Table 1. It is hoped that the detailed analysis of the constrained system dynamics \( \Sigma_\alpha \) here will provide some of the basic tools necessary for the nonstandard global analysis of large singularly perturbed systems of the form \( \Sigma_c \).

4. CONCLUSIONS

This paper summarizes the state space structure of a class of constrained singular systems and it is shown that the resulting dynamics is rich in structural details. The global structure if any in large singularly perturbed systems which include the jump phenomena needs to be explored. The parameter space structure can also be pursued and the implications of the singular surface and the Hurwitzian surface in the bifurcation analysis need investigation. The practical implications of the component instability and possible jump behavior in the network equations should be carefully analyzed for the large power system. A better understanding of the load behavior in the large sense is essential. It seems that rigorous energy function techniques for the transient stability analysis can be developed based on the stability boundary characterization. It is hoped that the detailed analysis of the constrained slow dynamics \( \Sigma_\alpha \) summarized here will aid in the development of an overall theory of the power system dynamics which covers global transient behavior such as the jump phenomena outside the constraint surface.

References


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