Classical Solution to Differential Equations
Arising in Circuit Analysis

1. Determine the initial current in each inductor and the initial voltage across each capacitor.

2. Write the integro-differential equations for the circuit using the mesh-current or node-voltage method.

   (a) It is likely to be helpful if first the network is simplified wherever possible, e.g., combine elements in series or parallel, come up with Thévenin equivalents, etc.

   (b) Differentiate if necessary to remove integral terms (we’re seeking a purely differential equation).

   (c) Combine the mesh or node equations algebraically to eliminate all but the desired variable (which will typically be a voltage or current).

Ultimately you will end up with an equation of the form:

\[ a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t) \]  

(1)

where all the \( a \)'s are constants and \( y(t) \) is the desired variable.

3. Using the results of part one and the mesh or node equations of part two, find all the necessary initial conditions (i.e., evaluate these equations at \( t = 0^+ \) to find the pertinent initial conditions for the desired variable). An \( n \)th-order differential equation requires \( n \) initial conditions:

\[ y(0^+), \ \frac{dy(0^+)}{dt}, \ \frac{d^2 y(0^+)}{dt^2}, \ \cdots, \ \frac{d^{n-1} y(0^+)}{dt^{n-1}}. \]  

(2)

4. If the forcing function \( f(t) \) is nonzero, find the particular solution \( y_p(t) \) (also known as the steady-state solution or the forced response) using the method of undetermined coefficients. This requires making an “educated guess” for \( y_p(t) \) based on the form of \( f(t) \). The “guess” will have a fixed functional form, but unknown coefficients. The coefficients are found by plugging \( y_p(t) \) into (1).

For example, consider the second-order equation

\[ a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = K \cos(\omega t) \]  

(3)
where the forcing function is \( K \cos(\omega t) \) and \( K \) is a constant. For a trigonometric forcing function, the particular solution (i.e., the educated guess) is of the form

\[
y_p(t) = A \sin(\omega t) + B \cos(\omega t).
\]  

(4)

The coefficients \( A \) and \( B \) are found by substituting (4) directly into (3) and equating the left and right sides. At first this may appear to be one equation and two unknowns, but two equations are produced by equating the coefficients of \( \sin \) and \( \cos \) on both sides (and since there is no \( \sin \) terms on the right hand side in this particular case, the sum of all the \( \sin \) coefficients on the left side must be zero). Thus we obtain (after swapping left and right sides)

\[
K \cos(\omega t) = -a_2 \omega^2 (A \sin(\omega t) + B \cos(\omega t)) + a_1 \omega (A \cos(\omega t) - B \sin(\omega t))
\]

\[
+ a_0 (A \sin(\omega t) + B \cos(\omega t))
\]  

(5)

\[
= \cos(\omega t) \left( -a_2 \omega^2 B + a_1 \omega A + a_0 B \right)
\]

\[
+ \sin(\omega t) \left( -a_2 \omega^2 A - a_1 \omega B + a_0 A \right).
\]  

(6)

Equating coefficients of the \( \cos \) and \( \sin \) functions yields

\[
K = A(a_1 \omega) + B(a_0 - a_2 \omega^2)
\]  

(7)

\[
0 = A(a_0 - a_2 \omega^2) - B(a_1 \omega).
\]  

(8)

These equations can be solved for \( A \) and \( B \):

\[
A = \frac{a_1 \omega K}{a_0^2 + (a_1^2 - 2a_0 a_2)\omega^2 + a_2^2 \omega^4},
\]  

(9)

\[
B = \frac{(a_0 - a_2 \omega^2)K}{a_0^2 + (a_1^2 - 2a_0 a_2)\omega^2 + a_2^2 \omega^4}.
\]  

(10)

Note that the particular solution is independent of the initial conditions. It should also be mentioned that one does not need to rely on making an educated guess for the particular solution—successive integration can be used to obtain it, but we leave that topic to others (see www.bruce-shapiro.com/math351/351pdf/chapters/chater4.pdf for an excellent discussion of the solution to linear differential equations with constant coefficients).

5. Obtain the homogeneous solution \( y_h(t) \) (also known as the natural response or transient solution) via the characteristic equation. The characteristic equation is obtained by assuming \( f(t) \) is zero and assuming a solution of the form \( y_h(t) = e^{st} \). Plugging in this assumed solution into (1) yields

\[
(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) e^{st} = 0.
\]  

(11)
Since the exponential is nonzero for all finite exponents, the only way this equation can be satisfied is if the term in parenthesis is zero. This is the characteristic equation for the differential equation:

\[ a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0. \] (12)

This can be factored producing \( n \) roots (some of which may be repeated, but we’ll save that topic for when we get to Laplace transforms). We label the roots of the equation \( r_i \) such that the equation can be written

\[ (s - r_{n-1})(s - r_{n-2}) \cdots (s - r_1)(s - r_0) = 0. \] (13)

Thus, any function of the form \( e^{r_i t} \) (where \( i \) goes from 0 to \( n - 1 \)) will be a solution to the homogeneous equation. The complete homogeneous solution consists of the sum of all the individual solutions:

\[ y_h(t) = C_0 e^{r_0 t} + C_1 e^{r_1 t} + \cdots + C_{n-1} e^{r_{n-1} t} = \sum_{i=0}^{n-1} C_i e^{r_i t} \] (14)

where the coefficients \( C_i \) are yet to be determined.

6. The complete solution is the sum of the particular and the homogeneous solution

\[ y(t) = y_p(t) + y_h(t). \] (15)

7. Evaluate the remaining constants \( C_i \) in \( y(t) \) using the initial conditions found in part three (you have \( n \) unknown coefficients and \( n \) initial conditions). Note that you have to evaluate these constants after adding the homogeneous and particular solutions (never before!).

**Example:** Consider the second-order differential equation given by

\[ \frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 3y(t) = 5 \cos(3t) \] (16)

with initial conditions \( y(0^+) = 11/6 \) and \( dy(0^+)/dt = 7 \). From part four we know the particular solution is given by

\[ y_p(t) = \frac{1}{3} \sin(3t) - \frac{1}{6} \cos(3t). \] (17)

The characteristic equation is

\[ s^2 + 4s + 3 = (s + 1)(s + 3) = 0. \] (18)

The roots are \(-1\) and \(-3\), giving a homogeneous solution of

\[ y_h(t) = C_0 e^{-t} + C_1 e^{-3t}. \] (19)
The total solution is therefore
\[ y(t) = y_p(t) + y_h(t) = \frac{1}{3} \sin(3t) - \frac{1}{6} \cos(3t) + C_0 e^{-t} + C_1 e^{-3t}. \] (20)

Using the initial conditions yields
\[ y(0^+) = \frac{11}{6} = -\frac{1}{6} + C_0 + C_1, \] (21)
\[ \frac{dy(0^+)}{dt} = 7 = 1 - C_0 - 3C_1. \] (22)

Equation (21) is obtained by equating the given initial condition with (20) with \( t \) set to zero while (22) is obtained by equating the other initial condition with the derivative of (20) evaluated at \( t = 0 \). Solving for \( C_0 \) and \( C_1 \) gives 6 and \(-4\), respectively. Thus the final solution is
\[ y(t) = \frac{1}{3} \sin(3t) - \frac{1}{6} \cos(3t) + 6e^{-t} - 4e^{-3t}. \] (23)