Example Solution to a Problem Involving a Second-Order Circuit

Consider the circuit shown below:

Given the initial current through the inductor \( i_l(t = 0^-) \) is zero and the initial voltage across the capacitor is \( v_c(t = 0^-) \) is 10 V, find the current \( i_2(t) \) for \( t \geq 0 \).

Differential Equation

After the switch has closed, the loop equation for the first loop is

\[
2i_1 + \frac{di_1}{dt} + 3 \int_0^t (i_1 - i_2)dt + v_c(0) = v_g. \tag{1}
\]

The loop equation for the second loop is

\[
3 \int_0^t (i_2 - i_1)dt - v_c(0) + 5i_2 = 0. \tag{2}
\]

Rearranging (2) yields

\[
3 \int_0^t (i_2 - i_1)dt + v_c(0) = 5i_2 \tag{3}
\]

Differentiating (3) yields

\[
3(i_1 - i_2) = \frac{di_2}{dt}. \tag{4}
\]

Solving for \( i_1 \) gives

\[
i_1 = i_2 + \frac{5}{3} \frac{di_2}{dt}. \tag{5}
\]

Equations (3) and (5) can be plugged into (1) to obtain a differential equation in \( i_2 \) alone:

\[
2 \left( i_2 + \frac{5}{3} \frac{di_2}{dt} \right) + \frac{d}{dt} \left( i_2 + \frac{5}{3} \frac{di_2}{dt} \right) + 5i_2 = v_g. \tag{6}
\]
Carrying out the differentiation and regrouping yields

\[
\frac{5}{3} \frac{d^2 i_2}{dt^2} + \frac{13}{3} \frac{di_2}{dt} + 7i_2 = v_g, \tag{7}
\]

which is equivalent to

\[
\frac{d^2 i_2}{dt^2} + \frac{13}{5} \frac{di_2}{dt} + \frac{21}{5} i_2 = \frac{3}{5} v_g. \tag{8}
\]

**Initial Conditions**

We are given the initial conditions for the inductor current (which is the same as \(i_1(0)\)) and the initial voltage across the capacitor, but are not told the initial conditions for the current \(i_2\). We must determine \(i_2(0)\) and \(di_2(0)/dt\) from the circuit and the given initial conditions.

The capacitor is in parallel with the 5\(\Omega\) resistor. Therefore the initial capacitor voltage is the same as the initial voltage across the resistor. Knowing the voltage across the resistor, we know its current:

\[
i_2(0) = \frac{v_c(0)}{5\Omega} = \frac{10 \text{ V}}{5\Omega} = 2 \text{ A}. \tag{9}
\]

For \(di_2(0)/dt\) we return to (4) and plug in the initial values for \(i_1\) (which is the same as \(i_t\)) and \(i_2\) (which was found in (9)):

\[
\frac{di_2(0)}{dt} = \frac{3}{5}(i_1(0) - i_2(0)) = \frac{3}{5}(0 - 2) = -\frac{6}{5} \text{ A}. \tag{10}
\]

**Particular Solution**

The particular solution depends on the form of the forcing function \(v_g(t)\). We will assume that \(v_g(t)\) is given by

\[
v_g(t) = 3t. \tag{11}
\]

For this forcing function the assumed particular solution is given by

\[
i_{2p}(t) = At + B \tag{12}
\]

where \(A\) and \(B\) are constants. To evaluate these constants, (12) is plugged into either (7) or (8). Plugging into (7) yields

\[
\frac{13}{3} A + 7(At + B) = 3t. \tag{13}
\]

Equating like powers of \(t\) on both sides produces two equations

\[
7A = 3, \tag{14}
\]
\[
\frac{13}{3} A + 7B = 0. \tag{15}
\]
From the first equation $A = \frac{3}{7}$. Plugging this into the second and solving for $B$ yields $B = -\frac{13}{49}$. Thus the particular solution is given by

$$i_{2p}(t) = \frac{3}{7} t - \frac{13}{49}.$$  

(16)

**Homogeneous Solution**

The homogeneous solution $i_{2h}(t)$ is obtained via the characteristic equation. To get the characteristic equation, assume the forcing function is zero and use an assumed solution of the form $\exp(st)$. Using (8) with $v_g$ set to zero yields:

$$(s^2 + \frac{13}{5}s + \frac{21}{5})e^{st} = 0.$$  

(17)

Since the exponential is never zero for a finite exponent, the only way this can be zero is if the term in parentheses, i.e., the characteristic equation, is zero. Factoring this equation we find the two roots are

$$s = -\frac{13}{10} \pm j\frac{\sqrt{251}}{10}.$$  

(18)

The complete homogeneous solution is given by summing the exponentials for each root and multiplying each term by an arbitrary constant (changes in magnitude do not affect a homogeneous solution—you get zero when you plug it into the differential equation no matter what the magnitude). The homogeneous solution is

$$i_{2h}(t) = C_1 e^{-\left(\frac{13}{10}j \frac{\sqrt{251}}{10}\right)t} + C_2 e^{-\left(\frac{13}{10} + j \frac{\sqrt{251}}{10}\right)t},$$

$$= e^{-\frac{13}{10}t} \left(C_1 e^{\frac{\sqrt{251}}{10}t} + C_2 e^{-\frac{\sqrt{251}}{10}t}\right).$$  

(19)

Using Euler’s formula ($\exp(\pm jx) = \cos(x) \pm j \sin(x)$) we can expand the exponentials with the imaginary exponent to obtain

$$i_{2h}(t) = e^{-\frac{13}{10}t} \left((C_1 + C_2) \cos\left(\frac{\sqrt{251}}{10}t\right) + j(C_1 - C_2) \sin\left(\frac{\sqrt{251}}{10}t\right)\right).$$  

(20)

Note that $C_1$ and $C_2$ are arbitrary constants. Thus $C_1 + C_2$ and $j(C_1 - C_2)$ are necessarily arbitrary constants themselves. We can identify the sum and difference of these constants with new arbitrary constants. Thus we write that the homogeneous solution is given by

$$i_{2h}(t) = e^{-\frac{13}{10}t} \left(C_a \cos\left(\frac{\sqrt{251}}{10}t\right) + C_b \sin\left(\frac{\sqrt{251}}{10}t\right)\right).$$  

(21)
where \( C_a \) and \( C_b \) are yet to be determined.

Note that if we really wanted to, we could have kept \( C_1 \) and \( C_2 \) in our equations. It will turn out that they are complex conjugates. Recall that the complex conjugate of a number \( z = x + jy \) has the opposite sign for the imaginary part. The complex conjugate is often indicated with an asterisk. Thus \( z^* = x - jy \). The sum of a number and its conjugate is twice the real part, i.e., \( z + z^* = 2x \), and \( j \) times the difference of a number and its conjugate is twice the imaginary part with a sign inversion, i.e., \( j(z - z^*) = -2y \). Therefore, because \( C_1 \) and \( C_2 \) are complex conjugates we are ensured that \( C_a \) and \( C_b \) are real numbers. By using \( C_a \) and \( C_b \) in (21) we avoid the hassle of messing with intermediate imaginary values which will ultimate cancel.

**Complete Solution**

The complete solution is the sum of the homogeneous and particular solutions:

\[
i_2(t) = i_{2p}(t) + i_{2h}(t),
\]

\[
i_2(t) = \frac{3}{7} t - \frac{13}{49} + e^{-\frac{13}{10}t} \left( C_a \cos \left( \frac{\sqrt{251}}{10} t \right) + C_b \sin \left( \frac{\sqrt{251}}{10} t \right) \right).
\]  

(22)

Once we have the complete solution (and not before!), we can evaluate the remaining constants using the initial conditions.

**Final Solution**

We now equate (9) with (22) evaluated at zero. This yields

\[
i_2(0) = 2,
\]

\[
= \frac{3}{7} 0 - \frac{13}{49} + e^{-\frac{13}{10}t} \left( C_a \cos \left( \frac{\sqrt{251}}{10} t \right) + C_b \sin \left( \frac{\sqrt{251}}{10} t \right) \right) \bigg|_{t=0},
\]

\[
= -\frac{13}{49} + C_a.
\]

From which we obtain

\[ C_a = \frac{111}{49}. \]  

(23)

Additionally, we equate (10) with the derivative of (22) evaluated at zero:

\[
\frac{di_2(t)}{dt} = -\frac{6}{5},
\]

\[
= \frac{3}{7} - \frac{13}{10} e^{-\frac{13}{10}t} \left( C_a \cos \left( \frac{\sqrt{251}}{10} t \right) + C_b \sin \left( \frac{\sqrt{251}}{10} t \right) \right)
\]

\[
+ \frac{\sqrt{251}}{10} e^{-\frac{13}{10}t} \left( -C_a \sin \left( \frac{\sqrt{251}}{10} t \right) + C_b \cos \left( \frac{\sqrt{251}}{10} t \right) \right) \bigg|_{t=0},
\]
\[ i_2(t) = \frac{3}{7} t - \frac{13}{49} + e^{-\frac{13}{16} t} \left( \frac{111}{49} \cos \left( \frac{\sqrt{251}}{10} t \right) + \frac{645}{49\sqrt{251}} \sin \left( \frac{\sqrt{251}}{10} t \right) \right). \]

When possible, it’s always a good idea to plot a solution to see if it is consistent with your understanding of what the solution should be. The following shows a plot (generated using Mathematica) of \( i_2(t) \) for the first seven seconds.

Note that at \( t = 0 \) the \( i_2 \) is 2A, which agrees with the initial condition. Furthermore, after the initial transient has died out, the curve appears to have be nearly a straight line with a slope of \( 3/7 \) (note that the value of \( i_2 \) at \( t = 7 \) should be approximate \( 3 - 13/49 \approx 2.73 \) and this agrees with the plot).