Using Mathematica and Matlab to Find Roots of Equations

Roots of a Polynomial

Mathematica

Assume we wish to find the roots of a polynomial

\[ a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \]  \hspace{1cm} (1)

The roots are the values of \( s \) which make the polynomial zero. In Mathematica we can use the command \texttt{Solve[\ldots]} to obtain the root. The command would appear like

\[
\text{Solve}\left[ an \ s^n + \ldots + a_2 \ s^2 + a_1 \ s + a_0 == 0, \{s\}\right]
\]

(where the \( n \) in \( s^n \) would have an integer value and the ellipsis would be filled in with some suitable terms). Note the double equal signs. As in most programming languages, a single equal sign is used as the assignment operator, i.e., it means to assign the value on the right of the equal sign to the variable on the left side. This statement says to find the values of \( s \) which make this statement true. In general the coefficients \( a_n \) will have numeric values. However, you can use symbolic values for the coefficients in polynomials up to fourth-order and Mathematica will provide the exact answer. For example, if you entered a quadratic equation

\[
\text{Solve}\left[ a_2 \ s^2 + a_1 \ s + a_0 == 0, \{s\}\right]
\]

\textbf{Mathematica} gives you a result that appears something like

\[
\left\{ \begin{align*}
\{ s & \to -a_1 - \sqrt{a_1^2 - 4 a_0 a_2} \\
\{ s & \to -a_1 + \sqrt{a_1^2 - 4 a_0 a_2} \}
\end{align*} \right. \right. \quad \left\{ \begin{align*}
\{ s & \to -a_1 - \frac{\sqrt{a_1^2 - 4 a_0 a_2}}{2 a_2} \\
\{ s & \to -a_1 + \frac{\sqrt{a_1^2 - 4 a_0 a_2}}{2 a_2} \}
\end{align*} \right. 
\]

This is the familiar quadratic equation. You will get a mess of an answer if you enter

\[
\text{Solve}\left[ a_3 \ s^3 + a_2 \ s^2 + a_1 \ s + a_0 == 0, \{s\}\right]
\]
but it is the exact answer. If you are interested in more details about the exact solution to the cubic equation you can go to mathworld.wolfram.com/CubicEquation.html.

Generally we will be interested in solutions to polynomials which have numeric values for the coefficients. As an example, we might enter the following in Mathematica:

```mathematica
Solve[24.1 s^3 - 13.2 s^2 + 6.0 s + 7.5 == 0, {s}]
```

This produces output which appears something like

\[
\{\{s \rightarrow -0.447868\}, \{s \rightarrow 0.497793 - 0.668623 i\}, \{s \rightarrow 0.497793 + 0.668623 i\}\}
\]

where \(i = \sqrt{-1}\). The right-arrows that appear in the Mathematica results are “rules.” If you subsequently want to use the numeric values, there are a number of ways to get to them. One thing you could do is issue a command such as

```mathematica
answer = s /. Solve[24.1 s^3 - 13.2 s^2 + 6.0 s + 7.5 == 0, {s}]
```

This creates an array, called `answer`, which holds the numeric roots. Hence the first element of `answer` is \(-0.447868\), the second element is \(0.497793 - 0.668623\)i, and so forth. (Array elements in Mathematica are accessed with an index in double brackets, e.g., `answer[[1]]` corresponds to the first element.)

**Matlab**

In Matlab polynomials are represented by arrays which hold the \(a_n\) coefficients in descending order. Thus, the polynomial \(24.1s^3 - 13.2s^2 + 6.0s + 7.5\) would be represented as the array \([24.1, -13.2, 6.0, 7.5]\). The roots of a polynomial can be obtained with the command `roots()`. For example:

```matlab
>> roots([24.1, -13.2, 6.0, 7.5])
```

```matlab
ans =

0.4978 + 0.6686i
0.4978 - 0.6686i
-0.4479
```
Partial Fraction Expansions

Mathematica

In Mathematica the command `Apart[]` can be used to obtain the partial-fraction expansion of a ratio of polynomials. For example, consider the polynomial

\[ F(s) = \frac{3.6s - 2.0}{24.1s^3 - 13.2s^2 + 6.0s + 7.5}. \]  

The expansion could be obtained as follows

\[
\text{Apart}[(3.6 \, s - 2.0)/(24.1 \, s^3 - 13.2 \, s^2 + 6.0 \, s + 7.5)]
\]

This produces the output

\[
-0.111746 + \frac{-0.0119233 + 0.111746 \, s}{0.447868 + s} + \frac{-0.0119233 - 0.111746 \, s}{0.694854 - 0.995586 \, s + s^2}
\]

Note that Mathematica does not break things down beyond the point where they are real—the second term contains a second-order equation in the denominator (which has complex roots). When trying to do inverse Laplace transforms using partial-fraction expansion, this is not necessarily a good thing since you cannot directly use Equation (12.64) from the text. To use (12.64) you need to have the expansion in fully factored form. However, you can work around this, as discussed in the Appendix.

Matlab

In Matlab the command to perform partial-fraction expansions is `residue()`. The numerator and denominator polynomials are both represented as arrays as discussed previously. So, continuing with the same example as before where the function is given in (2), in Matlab we could write

\[
\text{>> residue([3.6, -2.0],[24.1, -13.2, 6.0, 7.5])}
\]

\[
\text{ans =}
\]

\[
0.0559 - 0.0327i
0.0559 + 0.0327i
-0.1117
\]

But what does this mean? We can get more helpful output from `residue()` if we add more terms on the left, like so:
>> [r,p,k] = residue([3.6, -2.0],[24.1, -13.2, 6.0, 7.5])

\[
\begin{align*}
r &= \\
&= 0.0559 - 0.0327i \\
&= 0.0559 + 0.0327i \\
&= -0.1117 \\
p &= \\
&= 0.4978 + 0.6686i \\
&= 0.4978 - 0.6686i \\
&= -0.4479 \\
k &= \\
&= []
\end{align*}
\]

Here \( r \), which corresponds to the original output, are the coefficients of the partial-fraction-expansion terms, while \( p \) gives the corresponding pole for each term. The \( k \) vector will be non-null when you enter an improper rational function (since the example is a proper rational function the vector is null). This Matlab output says the partial-fraction expansion of (2) is
\[
\frac{0.0559 - j0.0327}{s - 0.4978 - j0.6686} + \frac{0.0559 + j0.0327}{s - 0.4978 + j0.6686} - \frac{0.1117}{s + 0.4479}. \tag{4}
\]

Appendix

In partial-fraction expansions, Mathematica will leave terms in quadratic form. Note that the second term in (3) is of the form
\[
\frac{bs + c}{s^2 + ds + h} \tag{5}
\]
where \( b, c, d, \) and \( h \) are constants. There are several ways to handle this (some arguably easier than what is presented here) but we will consider working through the general case.

Although we have not shown it in class, the Laplace transforms of the damped sine and damped cosine are
\[
\mathcal{L} \left[ e^{-at} \sin(\omega t) \right] = \frac{\omega}{(s + a)^2 + \omega^2}, \tag{6}
\]
\[ \mathcal{L} \left[ e^{-at} \cos(\omega t) \right] = \frac{s + a}{(s + a)^2 + \omega^2}. \]  

(7)

Equation (5) can be manipulated so that it is in the form of the transform of damped trig functions. We first get the denominator in the necessary form by completing the square as shown in the following:

\[
\frac{bs + c}{s^2 + ds + h} = \frac{bs + c}{s^2 + ds + \frac{d^2}{4} - \frac{d^2}{4} + h} \\
= \frac{bs + c}{(s + \frac{d}{2})^2 + h - \frac{d^2}{4}}.
\]

(8)

(9)

Comparing the denominator of this with the denominator of (7), we equate \(a\) with \(d/2\) and \(\omega^2\) with \(h - d^2/4\). The numerator also has to be manipulated so that we either have a constant (some multiple of \(\omega\)) or the product of a constant and \((s + a)\). Currently we have have neither of those. Thus, keeping in mind that \(a = d/2\), we write

\[
\frac{bs + c}{(s + \frac{d}{2})^2 + h - \frac{d^2}{4}} = \frac{bs + c}{s + 
\]

(10)

(11)

(12)

The first term is now in the form of a damped cosine with coefficient \(b\), exponent \(d/2\), and frequency \(\sqrt{h - d^2/4}\). The second term is almost in the form of a damped sinusoid. To get it the rest of the way, keeping in mind that \(\omega = \sqrt{h - d^2/4}\), we write

\[
b\frac{\frac{c}{b} - \frac{d}{2}}{(s + \frac{d}{2})^2 + h - \frac{d^2}{4}} = \frac{c - bd}{4h - d^2} \frac{\sqrt{h - d^2/4}}{\sqrt{h - d^2/4}}. 
\]

(13)

This, finally, is in form of a damped sinusoid with coefficient \((c - bd/2)/\sqrt{h - d^2/4} = (2c - bd)/\sqrt{4h - d^2}\), exponent \(d/2\), and frequency \(\sqrt{h - d^2/4}\).

The bottom line is the following

\[ \mathcal{L}^{-1} \left[ \frac{bs + c}{s^2 + ds + h} \right] = e^{-dt/2} \left\{ b \cos \left( \left[ h - \frac{d^2}{4} \right] \frac{1}{2} t \right) + \frac{2c - bd}{\sqrt{4h - d^2}} \sin \left( \left[ h - \frac{d^2}{4} \right] \frac{1}{2} t \right) \right\}. \]

(14)

The trig functions can be combined to give a single cosine term with a different amplitude and a phase offset in the argument of the cosine, but we will not bother with that here.
It may be of some interest (but don’t bother thinking about this too much) that Mathematica has a command \texttt{Residue[]} which is arguably similar to Matlab’s. However, you will find in practice that it is a bit tricky to use and can give misleading results. For example, this should give the same results as Matlab but instead produced all zeros:

\begin{verbatim}
fs = (3.6 s - 2.0)/(24.1 s^3 - 13.2 s^2 + 6.0 s + 7.5);
Table[
  Residue[fs,
    {s, (s /. Solve[Denominator[fs] == 0, {s}])[[i]]}],
  {i, 1, 3}]
\end{verbatim}