Chapter 4

Scaling FDTD Simulations to Any Frequency

4.1 Introduction

The FDTD method requires the discretization of time and space. Samples in time are $\Delta t$ apart whereas, in simulations with one spatial dimension, samples in space are $\Delta x$ apart. It thus appears that one must specify $\Delta t$ and $\Delta x$ in order to perform a simulation. However, as shown in Sec. 3.3, it is possible to write the coefficients $\Delta t/\varepsilon \Delta x$ and $\Delta t/\mu \Delta x$ in terms of the material parameters and the Courant number (ref. (3.19) and (3.20)). Since the Courant number contains the ratio of the temporal step to the spatial step, it allows one to avoid explicitly stating a definite temporal or spatial step—all that matters is their ratio. This chapter continues to examine ways in which FDTD simulations can be treated as generic simulations which can be scaled to any frequency. As will be shown, the important factors which dictate the behavior of the fields in a simulation are the Courant number and the points per wavelength for any given frequency.

4.2 Sources

4.2.1 Gaussian Pulse

In the previous chapter the source function, whether hardwired, additive, or incorporated in a TFSF formulation, was always a Gaussian. In the continuous world this function can be expressed as

$$f_g(t) = e^{-\left(\frac{t-d_g}{w_g}\right)^2} \tag{4.1}$$

where $d_g$ is the temporal delay and $w_g$ is a pulse-width parameter. The Gaussian has its peak value at $t = d_g$ and has a value of $e^{-1}$ when $t = d_g \pm w_g$. Since (4.1) is only a function of time, i.e., a function of $q \Delta t$ in the discretized world, it again appears as if the temporal step $\Delta t$ must be given explicitly. However, if one specifies the delay and pulse width in terms of temporal steps, the term $\Delta t$ appears in both the numerator and the denominator of the exponent. For example, in the last
chapter \(d_g\) was \(30 \Delta t\) and \(w_g\) was \(10 \Delta t\) so the source function could be written

\[
f_g(q \Delta t) = f_g[q] = e^{-\left(\frac{2 \pi \cdot 30}{100}\right)^2}.
\] (4.2)

Note that \(\Delta t\) does not appear on the right-hand side.

The discretized version of a functions \(f(q \Delta t)\) will be written \(f[n]\), i.e., the temporal step will be dropped from the argument since it does not appear explicitly in the expression for the function itself. The key to being able to discard \(\Delta t\) from the source function was the fact that the source parameters were expressed in terms of the number of temporal steps.

### 4.2.2 Harmonic Sources

For a harmonic source, such as

\[
f_h = \cos(\omega t),
\] (4.3)

there is no explicit numerator and denominator in the argument. Replacing \(t\) with \(q \Delta t\) it again appears as if the temporal step must be given explicitly. However, keep in mind that with electromagnetic fields there is an explicit relationship between frequency and wavelength. For a plane wave propagating in free space, the wavelength \(\lambda\) and frequency \(f\) are related by

\[
f \lambda = c \quad \Rightarrow \quad f = \frac{c}{\lambda}.
\] (4.4)

Thus the argument \(\omega t\) (i.e., \(2\pi f t\)) can also be written as

\[
\omega t = \frac{2\pi ct}{\lambda}.
\] (4.5)

For a given frequency, the wavelength is a fixed length. Being a length, it can be expressed in terms of the spatial step, i.e.,

\[
\lambda = N_\lambda \Delta x
\] (4.6)

where \(N_\lambda\) is the number of points per wavelength. This does not need to be an integer.

By relating frequency to wavelength and the wavelength to \(N_\lambda\), the discretized version of the harmonic function can be written

\[
f_h(q \Delta t) = \cos\left(\frac{2\pi c}{N_\lambda \Delta x} q \Delta t\right) = \cos\left(\frac{2\pi c \Delta t}{N_\lambda \Delta x} q\right),
\] (4.7)

or, simply,

\[
f_h[q] = \cos\left(\frac{2\pi S_c}{N_\lambda} q\right).
\] (4.8)

The expression on the right now contains the Courant number and the parameter \(N_\lambda\). In this form one does not need to state an explicit value for the temporal step. Rather, one specifies the Courant number \(S_c\) and the number of spatial steps per wavelength \(N_\lambda\). Note that \(N_\lambda\) will always be defined in terms of the number of spatial steps per the wavelength in free space. Furthermore, the wavelength is the one in the continuous world—we will see later that the wavelength in the FDTD grid is not always the same.
The period for a harmonic function is the inverse of the frequency

\[ T = \frac{1}{f} = \frac{\lambda}{c} = \frac{N_\lambda \Delta x}{c}. \]  

(4.9)

The number of time steps in a period is thus given by

\[ \frac{T}{\Delta t} = \frac{N_\lambda \Delta x}{c \Delta t} = \frac{N_\lambda}{S_c}. \]  

(4.10)

A harmonic wave traveling in the positive \( x \) direction is given by

\[ f_h(x, t) = \cos(\omega t - kx) = \cos\left(\omega \left(t - \frac{k}{\omega}x\right)\right). \]  

(4.11)

Since \( k = \omega \sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r} = \omega \sqrt{\mu_r \varepsilon_r / c} \), the argument can be written

\[ \omega \left(t - \frac{k}{\omega}x\right) = \omega \left(t - \frac{\sqrt{\mu_r \varepsilon_r}}{c}x\right). \]  

(4.12)

Expressing all quantities in terms of the discrete values which pertain in the FDTD grid yields

\[ \omega \left(t - \frac{c}{\sqrt{\mu_r \varepsilon_r}}x\right) = \frac{2\pi c}{N_\lambda \Delta x} \left(q \Delta t - \frac{\sqrt{\mu_r \varepsilon_r}}{c} m \Delta x\right) = \frac{2\pi}{N_\lambda} (S_c q - \sqrt{\mu_r \varepsilon_r} m). \]  

(4.13)

Therefore the discretized form of (4.11) is given by

\[ f_h[m, q] = \cos\left(\frac{2\pi}{N_\lambda} (S_c q - \sqrt{\mu_r \varepsilon_r} m)\right). \]  

(4.14)

This equation could now be used as the source in a total-field/scattered-field implementation. (However, note that when the temporal and spatial indices are zero this source has a value of unity. If that is the initial “turn-on” value of the source, that may cause artifacts which are undesirable. This will be discussed further when dispersion is discussed. It is generally better to ramp the source up gradually. A simple improvement is offered by using a sine function instead of a cosine since sine is initially zero.)

### 4.2.3 The Ricker Wavelet

One of the features of the FDTD technique is that it allows the modeling of a broad range of frequencies using a single simulation. Therefore it is generally advantageous to use pulsed sources—which introduce a broad range of frequencies—rather than a harmonic source. The Gaussian pulse is potentially an acceptable source except that it contains a dc component. In fact, in a Gaussian pulse dc is the frequency with the greatest energy. Generally one would not use the FDTD technique to model dc fields. Sources with dc components also have the possibility of introducing artifacts which are not physical (e.g., charges which sit in the grid). Therefore we consider a different pulsed source which has no dc component and which can have its most energetic frequency set to whatever frequency is desired.
The Ricker wavelet is equivalent to the second derivative of a Gaussian; it is simple to implement; it has no dc component; and, its spectral content is fixed by a single parameter. The Ricker wavelet is typically written

\[ f_r(t) = \left(1 - 2\pi^2 f_P^2 (t - d_r)^2\right) \exp\left[-\pi^2 f_P^2 (t - d_r)^2\right] \]  

(4.15)

where \( f_P \) is the “peak frequency” and \( d_r \) is the temporal delay. As will be more clear when the spectral representation of the function is shown below, the peak frequency is the frequency with the greatest spectral content.

The delay \( d_r \) can be set to any desired amount, but it is convenient to express it as a multiple of \( 1/f_P \), i.e.,

\[ d_r = M_d \frac{1}{f_P} \]  

(4.16)

where \( M_d \) is the delay multiple (which need not be an integer). An FDTD simulation is typically assumed to start at \( t = 0 \), but \( f_r(t) \) is not zero for \( t < 0 \)—rather \( f_r(t) \) asymptotically approaches zero for large and small values of the argument, but never actually reaches zero (other than at two discrete zero-crossings). However, with a delay of \( d_r = 1/f_P \) (i.e., \( M_d = 1 \)), \( |f_r(t < 0)| \) is bound by 0.001, which is small compared to the peak value of unity. Thus, the transient caused by “switching on” \( f_r(t) \) at \( t = 0 \) is relatively small with this amount of delay. Said another way, since the magnitude of \( f_r(t) \) is small for \( t < 0 \), these values can be approximated by assuming they are zero. For situations that may demand a smoother transition (i.e., a smaller initial turn-on value), the bound on \( |f_r(t < 0)| \) can be made arbitrarily small by increasing \( d_r \). For example, with a delay multiple \( M_d \) of 2, \( |f_r(t < 0)| \) is bound by \( 10^{-15} \).

The Fourier transform of (4.15) is

\[ F_r(\omega) = -\frac{2}{f_P \sqrt{\pi}} \left(\frac{\omega}{2\pi f_P}\right)^2 \exp\left[-jd_r\omega - \left(\frac{\omega}{2\pi f_P}\right)^2\right]. \]  

(4.17)

Note that the delay \( d_r \) only appears as the imaginary part of the exponent. Thus it affects only the phase of \( F_r(\omega) \), not the magnitude.

The functions \( f_r(t) \) and \( |F_r(\omega)| \) are shown in Fig. 4.1. For the sake of illustration, \( f_P \) is arbitrarily chosen to be 1 Hz. Different values of \( f_P \) change the horizontal scale but they do not change the general shape of the curve. To obtain unit amplitude at the peak frequency, \( F_r(\omega) \) has been scaled by \( f_P e^{\sqrt{\pi}/2} \).

The peak frequency \( f_P \) has a corresponding wavelength \( \lambda_P \). This wavelength can be expressed in terms of the spatial step such that \( \lambda_P = N_P \Delta x \), where \( N_P \) does not need to be an integer. Thus

\[ f_P = \frac{c}{\lambda_P} = \frac{c}{N_P \Delta x} \]  

(4.18)

The Courant number \( S_c \) is \( c\Delta t/\Delta x \) so the spatial step can be expressed as \( \Delta x = c\Delta t/S_c \). Using this in (4.18) yields

\[ f_P = \frac{S_c}{N_P \Delta t}. \]  

(4.19)

The delay can thus be expressed as

\[ d_r = M_d \frac{1}{f_P} = M_d \frac{N_P \Delta t}{S_c}. \]  

(4.20)
Figure 4.1: Normalized spectrum of the Ricker wavelet with $f_P = 1$ Hz. The corresponding temporal form $f_r(t)$ is shown in the inset box. For other values of $f_P$, the horizontal axis in the time domain is scaled by $1/f_P$. For example, if $f_P$ were 1 MHz, the peak would occur at 1 μs rather than at 1 s. In the spectral domain, the horizontal axis is directly scaled by $f_P$ so that if $f_P$ were 1 MHz, the peak would occur at 1 MHz.

Letting time $t$ be $q\Delta t$ and expressing $f_P$ and $d_r$ as in (4.19) and (4.20), the discrete form of (4.15) can be written as

$$f_r[q] = \left(1 - 2\pi^2 \left(\frac{S_c q}{N_P} - M_d\right)^2\right) \exp \left[-\pi^2 \left(\frac{S_c q}{N_P} - M_d\right)^2\right].$$  \hspace{1cm} (4.21)

Note that the parameters that specify $f_r[n]$ are the Courant number $S_c$, the points per wavelength at the peak frequency $N_P$, and the delay multiple $M_d$—there is no $\Delta t$ in (4.21). This function appears to be independent of the temporal and spatial steps, but it does depend on their ratio via the Courant number $S_c$.

Equation (4.15) gives the Ricker wavelet as a function of only time. However, as was discussed in Sec. 3.9, when implementing a total-field/scattered-field boundary, it is necessary to parameterize an incident field in both time and space. As mentioned previously and as shown in Appendix C, one can always obtain a traveling plane-wave solution to the wave equation simply by tweaking the argument of any function that is twice differentiable. Given the Ricker wavelet $f_r(t)$, $f_r(t \pm x/c)$ is a solution to the wave equation where $c$ is the speed of propagation. The plus sign corresponds to a wave traveling in the negative $x$ direction and the negative sign corresponds to a wave traveling in the positive $x$ direction. (Only 1D propagation will be considered here, but this type of tweaking can also be done in 2D and 3D.) Therefore, a traveling Ricker wavelet can be constructed by replacing the argument $t$ in (4.15) with $t \pm x/c$. The value of the function now depends on both
time and location, i.e., it is a function of two variables:

\[
fr(t \pm x/c) = fr(t, x), \\
= \left(1 - 2\pi^2 f_p^2 \left(t \pm \frac{x}{c} - d_r\right)^2\right) \exp \left[-\pi^2 f_p^2 \left(t \pm \frac{x}{c} - d_r\right)^2\right].
\] (4.22)

As before, (4.20) and (4.19) can be used to rewrite \(d_r\) and \(f_p\) in terms of the Courant number, the points per wavelength at the peak frequency, the temporal step, and the delay multiple. Replacing \(t\) with \(q\Delta t\), \(x\) with \(m\Delta x\), and employing the identity \(x/c = m\Delta x/c = m\Delta t/S_c\), yields

\[
fr[q, m] = \left(1 - 2\pi^2 \left(\frac{S_c q \pm m}{N_p} - M_d\right)^2\right) \exp \left[-\pi^2 \left(\frac{S_c q \pm m}{N_p} - M_d\right)^2\right].
\] (4.23)

This gives the value of the Ricker wavelet at temporal index \(q\) and spatial index \(m\). Note that when \(m\) is zero (4.23) reduces to (4.21).

### 4.3 Mapping Frequencies to Discrete Fourier Transforms

Assume the field was recorded during an FDTD simulation and then the recorded field was transformed to the frequency domain via a discrete Fourier transform. The discrete transform will yield a set of complex numbers that represent the amplitude of discrete spectral components. The question naturally arises: what is the correspondence between the indices of the transformed set and the actual frequency?

In any simulation, the highest frequency \(f_{\text{max}}\) that can exist is the inverse of the shortest period which can exist. In a discrete simulation one must have at least two samples per period. Since the time samples are \(\Delta t\) apart, the shortest possible period is \(2\Delta t\). Therefore

\[
f_{\text{max}} = \frac{1}{2\Delta t}.
\] (4.24)

The change in frequency from one discrete frequency to the next is the spectral resolution \(\Delta f\). It is dictated by the total number of samples which we will call \(N_T\) (an integer value). In general \(N_T\) would correspond to the number of time steps in an FDTD simulation. The spectral resolution is given by

\[
\Delta f = \frac{f_{\text{max}}}{N_T/2}
\] (4.25)

Plugging (4.24) into (4.25) yields

\[
\Delta f = \frac{1}{N_T\Delta t}.
\] (4.26)

In Sec. 4.2.2 it was shown that a given frequency \(f\) could be written \(c/\lambda = c/N_\lambda \Delta x\) where \(N_\lambda\) was the number of points per wavelength (for the free-space wavelength). After transforming to the spectral domain, this frequency would have a corresponding index given by

\[
N_{\text{freq}} = \frac{f}{\Delta f} = \frac{N_T c \Delta t}{N_\lambda \Delta x} = \frac{N_T c \Delta t}{N_\lambda \Delta x} = \frac{N_T}{N_\lambda} S_c.
\] (4.27)
Thus, the spectral index is dictated by the duration of the simulation $N_T$, the Courant number $S_c$, and the points per wavelength $N_\lambda$.

Note that in practice different software packages may index things differently. The most typical practice is to have the first element in the spectral array correspond to dc, the next $N_T/2$ elements correspond to the positive frequencies, and then the next $(N_T/2) - 1$ elements correspond to the negative frequencies (which will always be the complex conjugates of the positive frequencies in any real FDTD simulation). The negative frequencies typically are stored from highest frequency to lowest (i.e., fastest varying to slowest) so that the last value in the array corresponds to the negative frequency closest to dc $(-\Delta f)$.

From (4.19), the most energetic frequency in a Ricker wavelet can be written

$$ f_P = \frac{S_c}{N_P \Delta t}. $$

The spectral index corresponding to this is given by

$$ N_{freq} = \frac{f_P}{\Delta f} = \frac{N_T}{N_P} S_c. $$

This is identical to (4.27) except the generic value $N_{\lambda}$ has been replaced by $N_P$ which is the points per wavelength at the peak frequency.

## 4.4 Conductivity

When a material is lossless, the phase constant $k$ for a harmonic plane wave is given by $\omega \sqrt{\mu \varepsilon}$ and the spatial dependence is given by $\exp(\pm jkx)$. When the material has a non-zero electrical conductivity ($\sigma \neq 0$), the material is lossy and the wave experiences exponential decay as it propagates. The spatial dependence is given by $\exp(\pm \gamma x)$ where

$$ \gamma = j \omega \sqrt{\mu \varepsilon \left(1 - j \frac{\sigma}{\omega \varepsilon}\right)} = \alpha + jk $$

(4.30)

where $\alpha$ (the real part of $\gamma$) is the attenuation constant and $k$ (the imaginary part of $\gamma$) is the phase constant. The attenuation and phase constants can be expressed directly in terms of the material parameters and the frequency:

$$ \alpha = \frac{\omega \sqrt{\mu \varepsilon}}{\sqrt{2}} \left(1 + \left(\frac{\sigma}{\omega \varepsilon}\right)^2\right)^{1/2} - 1 \right)^{1/2}, $$

(4.31)

$$ k = \frac{\omega \sqrt{\mu \varepsilon}}{\sqrt{2}} \left(1 + \left(\frac{\sigma}{\omega \varepsilon}\right)^2\right)^{1/2} + 1 \right)^{1/2}. $$

(4.32)

When the conductivity is zero, the attenuation constant is zero and the phase constant reduces to that of the lossless case, i.e., $\gamma = jk = j\omega \sqrt{\mu \varepsilon}$.

Assume a wave is propagating in the positive $x$ direction in a material with non-zero electrical conductivity. The wave amplitude will decay as $\exp(-\alpha x)$. The skin depth $\delta_{\text{skin}}$ is the distance
over which the wave decays an amount $1/e$. Starting with a reference point of $x = 0$, the fields would have decayed an amount $1/e$ when $x$ is $1/\alpha$ (so that the exponent is simply $-1$). Thus, $\delta_{\text{skin}}$ is given by

$$\delta_{\text{skin}} = \frac{1}{\alpha}. \quad (4.33)$$

Since the skin depth is merely a distance, it can be expressed in terms of the spatial step, i.e.,

$$\delta_{\text{skin}} = \frac{1}{\alpha} = N_L \Delta x. \quad (4.34)$$

It is possible to use (4.31) to solve for the conductivity in terms of the attenuation constant. The resulting expression is

$$\sigma = \omega \epsilon \left[ \left( 1 + \frac{2\alpha^2}{\omega^2 \mu \epsilon} \right)^2 - 1 \right]^{1/2}. \quad (4.35)$$

As shown in Sec. 3.11, when the electrical conductivity is non-zero the electric-field update equation contains the term $\sigma \Delta t / 2 \epsilon$. Combining (4.35) with this term yields

$$\frac{\sigma \Delta t}{2 \epsilon} = \frac{\omega \Delta t}{2} \left[ \left( 1 + \frac{2\alpha^2}{\omega^2 \mu \epsilon} \right)^2 - 1 \right]^{1/2}. \quad (4.36)$$

Assume that one wants to obtain a certain skin depth (or decay rate) at a particular frequency which is discretized with $N_\lambda$ points per wavelength, i.e., $\omega = 2\pi f = 2\pi c / N_\lambda \Delta x$. Thus the term $\omega \Delta t / 2$ can be rewritten

$$\frac{\omega \Delta t}{2} = \frac{\pi \epsilon c \Delta t}{N_\lambda \Delta x} = \frac{\pi}{N_\lambda} S_c. \quad (4.37)$$

Similarly, using the same expression for $\omega$ and using $\alpha = 1 / N_L \Delta x$, one can write

$$2\alpha^2 / \omega^2 \mu \epsilon = \frac{2}{\left( 2\pi c / N_\lambda \Delta x \right)^2} \frac{\mu_0 \epsilon_0 \epsilon_r \mu_r}{\mu_0 \mu_r \epsilon_0 \epsilon_r} = \frac{N_\lambda^2}{2\pi^2 \epsilon_r \mu_r}. \quad (4.38)$$

Using (4.37) and (4.38) in (4.36) yields

$$\frac{\sigma \Delta t}{2 \epsilon} = \frac{\pi}{N_\lambda} S_c \left[ \left( 1 + \frac{N_\lambda^2}{2\pi^2 \epsilon_r \mu_r} \right)^2 - 1 \right]^{1/2}. \quad (4.39)$$

Note that neither the temporal nor the spatial steps appear in the right-hand side.

As an example how (4.39) can be used, assume that one wants a skin depth of $20 \Delta x$ for a wavelength of $40 \Delta x$. Thus $N_L = 20$ and $N_\lambda = 40$ and the skin depth is one half of the free-space wavelength. Further assume the Courant number $S_c$ is unity, $\epsilon_r = 4$, and $\mu_r = 1$. Plugging these values into (4.39) yields $\sigma \Delta t / 2 \epsilon = 0.0253146$.

A program with a TFSF boundary which introduces a sine wave with a frequency which is discretized at 40 points per wavelength is show in Program 4.1. This is similar to Program 3.7 except there is no magnetic loss term. Snapshots are taken every time step after the temporal index is within 60 steps from the end. The lossy layer starts at node 100.
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Program 4.1 Program to model a harmonic wave, introduced at a TFSF boundary, incident on a medium with non-zero electric conductivity.

```c
#include <stdio.h>
#include <math.h>

#define SIZE 200
#define LOSS 0.0253146 // loss factor sigma*delta_t/(2*epsilon)
#define PPW 40 // points per wavelength of inc. field
#define EPS_R 4.0 // relative permittivity of layer
#define LOSS_LAYER 100 // node where loss layer starts

int main()
{
    double ez[SIZE], hy[SIZE-1], eCoef1[SIZE], eCoef2[SIZE],
            hCoef1[SIZE-1], hCoef2[SIZE-1],
            char basename[80]="sim", filename[100];
    int frame=0,
    FILE *snapshot;

    /* initialize electric field */
    for (m=0; m<SIZE; m++)
        ez[m] = 0.0;

    /* initialize magnetic field */
    for (m=0; m<SIZE-1; m++)
        hy[m] = 0.0;

    /* set electric-field update coefficients */
    for (m=0; m<SIZE; m++)
        if (m<LOSS_LAYER) {
            eCoef1[m] = 1.0;
            eCoef2[m] = imp;
        } else {
            eCoef1[m] = (1.0-LOSS)/(1.0+LOSS);
            eCoef2[m] = imp/EPS_R/(1.0+LOSS);
        }

    /* set magnetic-field update coefficients */
    for (m=0; m<SIZE-1; m++) {
        hCoef1[m] = 1.0;
        hCoef2[m] = 1.0/imp;
    }

    /* do time stepping */
    for (qTime=0; qTime<maxTime; qTime++) {
        /* update magnetic field */
        for (m=0; m<SIZE-1; m++)
            hy[m] = hCoef1[m]*hy[m] + hCoef2[m]*(ez[m+1] - ez[m]);

        /* correction for Hy adjacent to TFSF boundary */
        hy[49] -= sin(2.0*M_PI/PPW*qTime)/imp;

        /* simple ABC for Ez adjacent to TFSF boundary */
        ez[0] = ez[1];

        /* update electric field */
        for (m=1; m<SIZE-1; m++)
            ez[m] = eCoef1[m]*ez[m] + eCoef2[m]*(hy[m] - hy[m-1]);

        /* correction for Ez adjacent to TFSF boundary */
        ez[50] += sin(2.0*M_PI/PPW*(qTime+0.5-(-0.5)));

        /* get snapshot if time near the end */
        if (qTime > maxTime-60) {
            char filename澳大=%s.%d", basename, frame++;
            FILE *snapshot=fopen(filename, "w");
            for (m=0; m<SIZE; m++)
                fprintf(snapshot, "%g\n", ez[m]);
            fclose(snapshot);
        }
    }
    return 0;
}
```

Figure 4.2: Maximum electric field which exists at each point for the snapshots generated by Program 4.1. The flat line over the first 50 nodes corresponds to the scattered field region. The reflected field travels without decay and hence yield the flat line. The total-field region between nodes 50 and 100 contains a standing-wave pattern caused by the interference of the incident and scattered fields. The exponential decay of the fields beyond node 100 which is where the lossy layer starts.

Figure 4.2 shows the maximum of the magnitude of the electric field as a function of position. For each position, all the snapshots were inspected and the maximum recorded. One can see the exponential decay starting at node 100. Between node 50 and node 100 there is a standing-wave pattern caused by the interference of the incident and reflected waves. From the start of the grid to node 50 the magnitude is flat. This is caused by the fact that there is only scattered field here—there is nothing to interfere with the reflected wave and we see that constant amplitude associated with a pure traveling wave. The ratio of the amplitude at nodes 120 and 100 was found to be 0.3644 whereas the ideal value of $1/e$ is 0.3679 (thus there is approximately a one percent error in this simulation).

If one were interested in non-zero magnetic conductivity $\sigma_m$, the loss term which appears in the magnetic-field update equations is $\sigma_m \Delta t / 2\mu$. This term can be handled in exactly the same way as the term resulting from electric conductivity.