Start reading Chapter 12 of *Computational Methods for Electromagnetics*.

The goal of this assignment is mainly to have you start using the computing tools which will be used throughout the semester. A fair amount of background material will be discussed prior to presenting the actual work that must be done.

In the finite-difference time-domain (FDTD) method, one must specify the behavior of a source as a function of time. One convenient function to employ is a Ricker wavelet. It is equivalent to the second derivative of a Gaussian; it is simple to implement; it has no dc component; and, its spectral content is fixed by a single parameter. The Ricker wavelet is typically written

\[ p(t) = \left(1 - 2\pi^2 f_P^2 (t - t_D)^2\right) \exp \left[-\frac{\pi^2 f_P^2 (t - t_D)^2}{2}\right] \]  

where \( f_P \) is the peak frequency and \( t_D \) is the temporal delay. As will be more clear when the spectral representation of the function is shown below, the “peak frequency” is the frequency with the greatest spectral content.

The delay \( t_D \) can be set to any desired amount, but it is convenient to express it as a multiple of \( 1/f_P \). For this assignment a value of \( t_D = 1/f_P \) will be used. An FDTD simulation is typically assumed to start at \( t = 0 \), but \( p(t) \) is not zero for \( t < 0 \)—in fact \( p(t) \) asymptotically approaches zero for large and small value of the argument, but never actually reaches zero. However, with a delay of \( t_D = 1/f_P \), \( |p(t < 0)| \) is bound by 0.001, which is small compared to the peak value of unity. Thus, the transient caused by “switching on” \( p(t) \) at \( t = 0 \) is relatively small. Said another way, since the magnitude of \( p(t) \) is small for \( t < 0 \), these values can be approximated by assuming there are zero. (For applications that may demand a smoother transition, the bound on \( |p(t < 0)| \) can be made arbitrarily small by increasing \( t_D \).)

The Fourier transform of (1) is

\[ P(\omega) = \frac{2}{f_P \sqrt{\pi}} \left(\frac{\omega}{2\pi f_P}\right)^2 \exp \left[-j t_D \omega - \left(\frac{\omega}{2\pi f_P}\right)^2\right] \]  

where \( f_P \) is the peak frequency and \( t_D \) is the temporal delay. As will be more clear when the spectral representation of the function is shown below, the “peak frequency” is the frequency with the greatest spectral content.
Figure 1: Normalized spectrum of the Ricker wavelet with $f_P = 1$ Hz. The corresponding temporal form $p(t)$ is shown in the inset box. For other values of $f_P$, the horizontal axis in the time domain is scaled by $1/f_P$. For example, if $f_P$ were 1 MHz, the peak would occur at 1 μs rather than at 1 s. In the spectral domain, the horizontal axis is directly scaled by $f_P$ so that if $f_P$ were 1 MHz, the peak would occur at 1 MHz.

Note that the delay $t_D$ only appears as the imaginary part of the exponent. Thus it affects the only the phase of $P(\omega)$, not the magnitude.

The functions $p(t)$ and $|P(\omega)|$ are shown in Fig. 1. For the sake of illustration, $f_P$ is arbitrarily chosen to be 1 Hz. Different values of $f_P$ change the horizontal scale but they do not change the general shape of the curve. To obtain unit amplitude at the peak frequency, $P(\omega)$ has been scaled by $f_P e^{\pi/2}$.

In the FDTD method both time and space are discretized. The fields are only defined at “snapshots” in time. The snapshots are calculated at discrete times which are $\Delta t$ apart. Similarly, the fields are sampled at spatial locations that are separated from their neighbors by some finite amount. The details of the FDTD method are not a concern at this point, so simply assume the spatial samples of the field are arranged on a Cartesian grid and the spacing between sample point, i.e., nodes, is $\delta$. The source function in an FDTD simulation is not continuous—instead a discrete value is used at each time step. These discrete values can be considered a discrete function which is obtained by
sampling a continuous function at given points. To that end we will consider the discrete function $p[n] = p(n\Delta t)$, where $n$ is an integer. Brackets are used for the arguments of discrete functions and parentheses for continuous ones.

In an FDTD simulation it is often the case that one wants to model an object which has specific physical dimensions. Additionally, there is often an excitation frequency or range of frequencies that are of interest. Thus one can specify how a particular simulation should be discretized in space and time in terms of meters and seconds. However, there is another way to specify the discretization of simulation. One can specify things in terms of the number of spatial samples per wavelength of the excitation and the ratio of the speed of light times the temporal step size (i.e., $c\Delta t$) to the spatial step size which we will call \textit{delta}. This way of thinking of things may be take some getting used to, but the advantage is that the results then can scale to any frequency provided the physical dimensions are similarly scaled.

Returning now to the function $p[n]$, we want to determine its values when we are told the number of sample per wavelength at the peak frequency and the ratio $c\Delta t/\delta$. Assuming a uniform spatial step of $\delta$, we let $s$ represent the “Courant number” which is defined as $s \equiv c\Delta t/\delta$. (Throughout the following discussion the speed of light and the wavelength are assumed to be the appropriate values for the point at which the field is introduced. For example, if the field is introduced in a dielectric with a relative permittivity of 9, the speed of light $c$ and wavelength are one-third what they would be in free space.) Further assume that the spatial step size is such that there are $N_P$ spatial steps at the wavelength $\lambda_P$ corresponding to the peak frequency of the Ricker wavelet, i.e., $\lambda_P = N_P\delta$. As will be shown later in the course, the value of $s$ is bounded by the stability limit (and one typically wants it to be as large as possible for reasons that should become evident later in the course). However, $N_P$ is \textit{chosen} to ensure that the peak frequency is sampled at $N_P$ points per wavelength. Writing $\delta = c\Delta t/s$ and $\lambda_P = c/f_P = c\tau_D$, one obtains

$$t_D = \frac{1}{f_P} = N_P\Delta t/s \quad (3)$$

Letting time $t$ be $n\Delta t$ and expressing $t_D$ and $f_P$ as in (3), the discrete form of (1) can be written as

$$p[n] = \left(1 - 2\pi^2 \left(\frac{sn}{N_P} - 1\right)^2\right) \exp\left[-\pi^2 \left(\frac{sn}{N_P} - 1\right)^2\right] \quad (4)$$

Note that the parameters that specify $p[n]$ are the Courant number $s$ and the points per wavelength at the peak frequency $N_P$—there is no $\Delta t$ in (4). This function appears to be independent of the temporal and spatial step sizes, but it does depend on their ratio via the Courant number $s$. 
With that as background material, do the following:

1. Write a program in a compiled language (i.e., C or FORTRAN) which prompts the user for
the parameters in (4), i.e., \( N_P \) and \( s \), as well as the total number of time steps. The values
should be written to an output file that the user also specifies.

2. Run the program you wrote for the first problem for 65536 time steps using an \( N_P \) of 40
and a Courant number, \( s \), of \( 1/\sqrt{3} \). Take the resulting data file and read it into either Matlab
or Mathematica. Take the (discrete) Fourier transform of the data and plot the real part, the
imaginary part, and the absolute value of the spectrum as a function of frequency. You do
not need to plot the entire spectrum but only the “interesting” part of it (i.e., you may ignore
the parts where the spectrum is effectively zero). Make sure your results are consistent with
the discussion given above. Is the peak where it should be? Make sure you provide evidence
that it is.

That concludes the work that must be done by all students. The following should be read by
everybody but only the 517 students need to do the problem at the end.

Equation (1) gives the Ricker wavelet as a function of only time. However, we are often interested
in traveling waves and hence the wave is parameterized by both time and space. One can always
obtain a traveling plane-wave solution to the wave equation simply by tweaking the argument of
any function that is twice differentiable. Given a function \( f(t) \), \( f(t \pm x/c) \) is a solution to the
wave equation where \( c \) is the speed of propagation (again, this is the speed of propagation for the
particular material in which the wave is propagating—it is not necessarily the free-space value of
\( 3 \times 10^8 \) m/s). The plus sign corresponds to a wave traveling in the negative \( x \) direction and the
negative sign corresponds to a wave traveling in the positive \( x \) direction. (Only 1D propagation
will be considered here, but this type of tweaking can also be done in 2D and 3D.) Therefore, a
traveling Ricker wavelet can be constructed by replacing the argument \( t \) in (1) with \( t \pm x/c \). The
value of the function now depends on both time and location, i.e., it is a function of two variables:

\[
p(t \pm x/c) = p(t,x), \quad n = \left(1 - 2\pi^2 f_p^2 \left(t \pm \frac{x}{c} - t_D\right)^2\right) \exp \left[-\pi^2 f_p^2 \left(t \pm \frac{x}{c} - t_D\right)^2\right]. \quad (5)
\]
As before, we can use (3) to rewrite $t_D$ and $f_P$ in terms of the Courant number, the points per wavelength at the peak frequency, and the temporal step size. Replacing $t$ with $n\Delta t$, $x$ with $i\delta$ ($i$ is an index for a node within the grid), and employing the identity $x/c = i\delta/c = i\Delta t/s$, yields

$$p[n, i] = \left( 1 - 2\pi^2 \left( \frac{sn \pm i}{N_P} - 1 \right)^2 \right) \exp \left[ -\pi^2 \left( \frac{sn \pm i}{N_P} - 1 \right)^2 \right].$$

(6)

This gives the value of the Ricker wavelet at time step $n$ and spatial location $i$. Note that when $i$ is zero (6) reduces to (4).

The following problem is for EE 517 students (and is optional/extra credit for 417 students).

3. Using only Matlab or Mathematica, write the code which implements (6). Use the negative sign for the $i$ values. Using the same parameters as the second problem but with an offset (or location) of 10 (i.e., $i = 10$), generate 65536 samples points. Again plot the spectrum and compare it to the spectrum obtained in the second problem.