Quick Method of Convolution for Piecewise Constant Functions

If \( x(t) \) and \( h(t) \) are piecewise constant functions (i.e., "box" functions or unions of box functions), then it is only necessary to compute the convolution integral at the end of each interval, and then "connect the dots" with straight lines. This is because \( y(t) = x(t) * h(t) \) is linear in every interval.

**Example:** Compute \( y(t) = x(t) * h(t) \), where \( x(t) \) and \( h(t) \) are shown below.

\[
\begin{array}{c|ccc|c}
0 & 1 & 2 & 3 & 4 \\
\hline
x(t) & 1 & 2 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc|c}
0 & 1 & 2 & 3 & 4 \\
\hline
h(t) & 1 & 1 & 1 & 0 \\
\end{array}
\]

**Interval 1:** \(-\infty < t < 0\)

End of Intervals:

\[
\begin{array}{c|ccc|c}
0 & 1 & 2 & 3 & 4 \\
\hline
x(t) & 1 & 1 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|ccc|c}
0 & 1 & 2 & 3 & 4 \\
\hline
h(t) & 1 & 1 & 1 & 0 \\
\end{array}
\]

No overlap, so \( w(t) = x(t) h(t-t) = 0 \) everywhere in

**Interval 1,**

Hence, \( y(t) = \int_{-\infty}^{0} w(t) \, dt = 0 \), **interval 1.**
Interval 2: \( 0 \leq t < 2 \)

\[ y(t) = \int_{-1}^{t} (2x+1) \, dx = \left[ x^2 + x \right]_{-1}^{t} = 2t^2 + t - 1 \]

Area under \( y(t) \) increases linearly with \( t \).

\[ h(t-1) \text{, in Interval 2} \]

\[ y(2) = 2 \times 2 \times (3-1) = 4 \]

Interval 2: \( -1 < t \leq 2 \)

Width of overlap:

\[ h(2-t) \]

Consider \( y(t) \) and \( y(t) \) with a straight line:

Interval 3: \( 2 \leq t < 4 \)

\[ y(t) = \int_{0}^{t} (2x+1) \, dx = \left[ x^2 + x \right]_{0}^{t} = t^2 + t \]

\[ h(t-2) \]

\[ y(3) = \int_{1}^{3} (2x+1) \, dx = 3 \]

\[ h(4-t) \]

\[ y(4) = \int_{2}^{4} (2x+1) \, dx = -4 \]
Determine $y(t)$ and $y(4)$ with a straight line.

**Interval 4:** $4 \leq t < 6$

Compute $y(6) = \int_{a}^{\infty} x(t) h(t-6) \, dt = \int_{a}^{4} 0 \, dt = 0$

(no overlap)

**Interval 5:** $t \geq 6$, $y(t) = 0$ no overlap.

Complete graph of $y(t)$:

Find $y(t)$ from sketch:

$$y(t) = \begin{cases} 
0, & t < 0 \\
2t, & 0 \leq t < 2 \\
-4t + 8, & 2 \leq t < 4 \\
2t - 12, & 4 \leq t < 6 \\
0, & t \geq 6
\end{cases}$$