1. KVL:
\[ V_c(s) = I(s)(Z_1(s) + Z_2(s)) - I(s)Z_0(s) \]
\[ 0 = -I(s)Z_2(s) + I(s)(Z_1(s) + Z_2(s) + Z_3(s)) \]

\[ \begin{bmatrix} Z_1 + Z_2 & -Z_1 \\ -Z_2 & Z_2 Z_3 + Z_3 \end{bmatrix} \begin{bmatrix} I(s) \\ V_c(s) \end{bmatrix} = \begin{bmatrix} 0 \\ V_i(s) \end{bmatrix} \]

\[ I_2(s) = \frac{V_c(s)}{-Z_2} = \frac{Z_2 V_c(s)}{-Z_2} \]
\[ H(s) = \frac{Z_2 T(s)}{Z_2 T(s) - Z_2} \]

2. a) Using
\[ Z_1 = \frac{1}{\omega_1}, \quad Z_2 = \frac{1}{\omega_2}, \quad Z_3 = \frac{1}{\omega_3}, \quad Z_4 = \frac{1}{\omega_4} \]
\[ H(s) = \frac{s + \frac{1}{\omega_3} + \frac{1}{\omega_4}}{s + \frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{\omega_3} + \frac{1}{\omega_4}} \]

Design Requirements:
- i) gain \( \geq 3 \) dB for \( f < 2 \) kHz
- ii) gain \( \leq -40 \) dB for \( f > 10 \) kHz

Design:
- Let the two pole frequencies be \( \omega_1, \omega_2 \), with \( \omega_1 < \omega_2 \).
  - From i), want \( \omega_1 > 2\pi \) kHz
  - ii) want \( \omega_2 > 2\pi \) kHz

Let
\[ \omega_1 = 2\pi \omega_{5000} = \frac{1}{\omega_{5000}}, \quad \omega_2 = 2\pi \omega_{4000} = \frac{1}{\omega_{4000}} \]

Want \( \frac{1}{\omega_1} \) very small compared to \( \frac{1}{\omega_2} \) and \( \frac{1}{\omega_3} \), so

The value of \( C \) is chosen to be
\[ C = \frac{1}{2 \pi f_{10 \text{kHz}}} \]

Somewhat arbitrarily, pick \( R_2 = 10 \) kΩ

The design requirements are met. Design is not unique.
Design requirements: Gain > -3 dB for frequencies less than 2 kHz (12,566 rad/s);
Gain < -40 dB for frequencies greater than 40 kHz (251,330 rad/s).

Write the transfer function as

$$H(s) = \frac{\frac{1}{R_1R_2C_1C_2}}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right)s + \frac{1}{R_1R_2C_1C_2}}$$

Selecting the poles at $p_1 = 2\pi 3000$, $p_2 = 2\pi 4000$ yields the Bode plot below, which satisfies the design specifications.

Finally, pick resistor and capacitor values so that the term $\frac{1}{R_2C_1}$ is negligible compared to $\frac{1}{R_1C_1}$, $\frac{1}{R_2C_2}$ and let $\frac{1}{R_1C_1} \sim p_1$, $\frac{1}{R_2C_2} \sim p_2$. The solution is not unique. One choice is $R_1 = 530 \, \Omega$, $C_1 = 0.1 \mu F$, $R_2 = 10^7 \, \Omega$, $C_2 = 3.98 \, pF$. The Bode plot for this design is shown below. The pole locations have changed from $s = -25,133$ and $s = -18,850$ for the figure above, to $s = -2.5130$ and $s = -1.8865$ for the figure below, resulting in a negligible change in the Bode plots.
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1. (continued) The design is not unique. As a candidate, let $C_1 = 10^4 \text{ F}$, so

then $\frac{1}{12C_1} = p = \frac{2}{2\pi \times 3000} \Rightarrow R_1 = \frac{1}{C_1 \frac{2\pi}{3000}} = 5305.72$

If $R_2 = 10R_1$, then $\frac{1}{R_2} = \frac{1}{10R_1} = 2.7 \times 10^3$

$\Rightarrow C_2 = \frac{1}{10^2 \times \frac{2\pi}{7000}} = 39.8 \times \frac{1}{R_2^2}$

Using these values, $\frac{1}{R_2} = \frac{1}{10^2 \times \frac{2\pi}{7000}} = 10$, which is small compared to $\frac{1}{p} = \frac{2}{2\pi \times 3000}$ and $\frac{1}{p} = 2.7 \times 10^3$

4. Cannot design a 2nd-order Butterworth LP filter because poles are real-valued.

Using Matlab (code appended) the Bode plot match the specifications.

2. Design a 2nd-order Butterworth filter to satisfy:

- gain $\geq 1 \text{ dB}$ for $0 \leq f \leq 2000 \text{ Hz}$
- gain $\leq -20 \text{ dB}$ for $f \geq 10 \text{ kHz}$

Since filter order is known to be $N = 2$, just select cutoff frequency $\omega_c$. Use $-10 \log (1 + (\frac{\omega_c}{\omega_0})^{2N}) = \text{gain in dB}$.

Solve for $\omega_c$ from $-10 \log \left(1 + \left(\frac{2\pi \times 2000}{\omega_c} \right)^4 \right) = -20 \text{ dB}$.

$\frac{2\pi \times 2000}{\sqrt{10}} = 25.13 \Rightarrow \omega_c = 2\pi \times 2000 = 17,616 \text{ rad/s} \approx (2,804 \text{ kHz}).$

So, pick $\omega_c = 20,000$.

$H(s) = \frac{\omega_c^2}{s^2 + 2s\omega_c + \omega_c^2}$ where $\omega_c = 20,000 \text{ rad/s}$

$\Rightarrow H(s) = \frac{1}{\sqrt{2}}$

The attached Matlab results show the specifications are satisfied.
Problem 2. Selecting $\omega_c = 20,000$ rad/s and damping constant $\xi = 1/\sqrt{2}$, then the Bode plot is shown below, clearly satisfying the specifications (gain $>-1$ dB for frequencies less than 2000 Hz (12,566 rad/s) and gain $<-40$ dB for frequencies larger than 40,000 Hz (251,330 rad/s)).

Problem 3, $N = 6$, $\omega_c = 26,000\pi$. Design requirements: Gain $>-3$ dB for frequencies less than 12 kHz (75,398 rad/s); Gain $<-40$ dB for frequencies greater than 20 kHz (125,660 rad/s).
Design a Butterworth filter \((\text{find } w_c, N)\) to satisfy:

1) Gain \(\geq -2\) for \(f \leq 12\,\text{kHz}\) \((24,000\,\text{rad/s})\),
2) Gain \(\leq -40\) for \(f \geq 18\,\text{kHz}\) \((36,000\,\text{rad/s})\).

\[-10\log \left(1 + \left(\frac{w_c}{w}\right)^2\right) = -2, \quad -10\log \left(\frac{N^2}{w_c^2}\right) = -40\]

\[
\begin{align*}
\frac{24,000\pi}{w_c} & = 2, \\
\frac{36,000\pi}{w_c} & = 2^{10}.
\end{align*}
\]

\[
\begin{align*}
\log \left(\frac{24,000\pi}{w_c}\right) & = \log \left(10^2\right), \\
\log \left(\frac{36,000\pi}{w_c}\right) & = 10^{10}.
\end{align*}
\]

\[
N = \frac{1}{2} \log \left(\frac{10^{10}}{10^2}\right) = 12.019 \quad \Rightarrow \quad \text{Round up to } N = 13.
\]

Using \(N = 13\),

\[
\frac{24,000\pi}{w_c} = 10^2 \Rightarrow w_c = \frac{24,000\pi}{10^2} = 76,970\,\text{rad/s} \quad \text{(at 12.5 kHz)}
\]

\[
\frac{36,000\pi}{w_c} = 10^{10} \Rightarrow w_c = \frac{36,000\pi}{10^{10}} = 7,936\,\text{rad/s} \quad \text{(at 12.5 kHz)}
\]

Any \(w_c\) satisfying

\[76,970 \leq w_c \leq 79,361\] can be selected with \(N = 13\).
(a) \[ H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{s^2 + \frac{1}{RC} s + \frac{1}{2C}} \]. No zeros. Poles at \[ s_{1,2} = -\frac{1}{2RC} \pm \frac{1}{2C} \].

(b) For real-valued poles, require \( \left(\frac{1}{2RC}\right)^2 \leq \frac{1}{2C} \). Solution is not unique, e.g., if \( L = 10 \text{ mH} \) then if \( C = 1 \text{ uF} \), \( R \) must be no larger than \( 50 \Omega \), so \( R = 47 \) is one choice. Then, poles at \( s = -7008.8 \text{ rad/s}, s = 14,267.8 \text{ rad/s} \).

(c) For repeated poles, require \( \frac{1}{2RC} = \frac{1}{\sqrt{2C}} \). So, if \( L = 10 \text{ mH}, C = 1 \text{ uF} \), then \( R = 50 \Omega \) \( \Rightarrow \) poles \( s = -10,000 \text{ rad/s} \).

(d) For complex poles, require \( \frac{1}{2RC} < \frac{1}{\sqrt{2C}} \), so for \( L = 10 \text{ mH}, C = 1 \text{ uF} \), any \( R > 50 \Omega \) will work. If \( R = 100 \text{ k\Omega} \), poles at \( s_1, s_2 = -5000 \pm j 666.25 \text{ rad/s} \).
(13.61) \[ H(s) = \frac{R}{s^2 + 2s + 1} \rightarrow h(t) = e^{-t} \]

Find \( y(t) = x(t) + u(t) \) for \( x(t) = u(t) - u(t-1) \) \( t \geq 0 \)

- \( y(t) = 0 \) for \( t < 0 \)
- \( y(t) = \int_0^t e^{-\tau} d\tau \) for \( 0 \leq t \leq 1 \)
- \( y(t) = \left(1 - e^{-(t-1)}\right) e^{-t} \) for \( t > 1 \), \( y(t) = \int_0^t e^{-\tau} d\tau = e^{-t} \) for \( t = 1 \)

(13.63) \[ y(t) = x(t) + x(t-30) \]

a) \[ y(t) = 0 \] for \( t > 0 \)
- \( y(t) = \int_0^{10} (1)(\tau) d\tau = 10 \) for \( 0 \leq t \leq 10 \)
- \( y(t) = \int_0^{40} (1)(\tau) d\tau = 40 \) for \( 10 \leq t \leq 40 \)
- \( y(t) = 0 \) for \( t > 40 \)

b) \[ y(t) = 0 \] for \( t > 0 \)
- \( y(t) = \int_0^{40} (4)(\tau) d\tau = 40 \) for \( 0 \leq t \leq 10 \)
- \( y(t) = \int_0^{10} (4)(\tau) d\tau = 400 \) for \( 10 \leq t \leq 40 \)
- \( y(t) = \int_0^{30} (4)(\tau) d\tau = 40(10 - (t-70)) \) for \( 40 \leq t \leq 50 \)
- \( y(t) = 0 \) for \( t > 50 \)
\[ h(t) = \begin{cases} 40 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ y(t) = \begin{cases} 0, & t < 0 \\ 400t, & 0 \leq t \leq 1 \\ 400, & 1 \leq t \leq 4 \\ 400(41-t), & 4 \leq t \leq 41 \\ 0, & t \geq 41 \end{cases} \]

e) yes, \( h(t) \) is approaching 40 s/m, so \( y(t) \) must approach 40 x(t).

\[ H(s) = \frac{16s}{4s + 4s + 16s} = \frac{0.8s}{s + 2} \]

\[ h(t) = 0.8sH(s) - 1.6e^{-2t} \]

\[ V(t) = h(t) \cdot 75 \text{ V} \]

\[ V(t) = \int_0^t (0.8s - 1.6e^{-2t}) 75 \, dt = 60 - 12e^{-2t} \]

\[ V(t) = 60 + 60(e^{-2t}) = 60e^{-2t} \text{ V} \]