Instructions: Closed book, 1 hour (50 minute) exam. Two pages of notes (two-sided) are allowed. Calculators are allowed. Problem solutions should include all work leading to the solution. Some useful formulae are appended.

1. (20 points) A discrete-time, linear, time-invariant system with input \( x(n) \) and output \( y(n) \) has the impulse response \( h(n) = [2, 2, 1, -1] \), where the underline denotes the \( n = 0 \) sample, and all signal values outside the range specified are zero.
   a. (2 points) Is the system causal? (You must provide correct justification to receive credit.)
   b. (3 points) Is the system BIBO stable? (You must provide correct justification to receive credit.)
   c. The system input is \( x(n) = [-1, -1, 0, 1, 1] \), where the underline denotes the \( n = 0 \) sample, and all signal values outside the range specified are zero.
   d. (2 points) \( y(n) \) begins at \( n = ? \)
   e. (3 points) \( y(n) \) ends at \( n = ? \)
   f. (10 points) Find \( y(n) \) for all \( n \).

\[
\begin{align*}
\text{a) not causal since } & h(1) = 2 \\
\text{b) is BIBO stable since } & \sum_{k=-\infty}^{\infty} |h(k)| = 6 < \infty \\
\text{c) begin at } & n = 2 + 2 = 4 \\
\text{end at } & n = 2 + 2 = 4 \\
\text{2.1)} y(n) = & [-2, -4, -3, 2, 5, 3, 0, -1] \\
& \text{with } n = 0 \\
& \begin{bmatrix}
-1 & 1 & 2 & 2 \\
\end{bmatrix}
\end{align*}
\]
2. (25 points) A periodic signal, $x(t)$, has period $T = \frac{\pi}{5}$ sec, fundamental frequency
\[ \omega_0 = \frac{2\pi}{T} = 10 \text{ rad/s}, \]
and exponential Fourier series coefficients
\[ c_k = \begin{cases} 
5, & k = 0 \\
10, & k = 1, -1 \\
-7j, & k = 2 \\
7j, & k = 2 \\
-3, & k = 3, -3 \\
0, & \text{otherwise}
\end{cases} \]

a. (5 points) Find the power in the signal $x(t)$.
b. (5 points) A truncated Fourier series is of the form
\[ x_N(t) = \sum_{k=-N}^{N} c_k e^{j2\pi kt/T}. \]

Find the normalized mean-square truncation error for
i. (3 points) $N = 1$.
ii. (2 points) $N = 5$.
c. (5 points) Accurately sketch the magnitude spectrum for $x(t)$, labeling all relevant amplitudes.
d. (10 points) The signal, $x(t)$, is applied as input to a linear, time-invariant system with output, $y(t)$. The system has impulse response $h(t)$ with Fourier transform (the system frequency response) $H(\omega)$, where $H(\omega) = \begin{cases} 20, & 5 \leq |\omega| \leq 25 \\
0, & \text{otherwise} \end{cases}$. Find the Fourier series for $y(t)$. Express $y(t)$ in simplest form (e.g., sines and cosines).

\[
\begin{align*}
\text{a)} & \quad P = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{5^2}{1^2} + (\frac{2}{1^2} + \frac{2}{1^2} + \frac{3^2}{1^2}) \times 2 = 25 + 2 \times 158 = 341 \\
\text{b)} & \quad \frac{P - (\sum_{k=\pm 1}^{\pm 2} c_k^2)}{P} = \frac{341 - (25 + 2 \times 100)}{341} = \frac{116}{341} \\
\text{c)} & \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\frac{\pi}{5}} = 10 \quad \text{rad/s} \\
\text{d)} & \quad y(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T} \\
\therefore y(t) & = 450 \cos(200t) + 280 \sin(200t) \]
3. (30 points) A linear, time-invariant (LTI) system with overall impulse response \( h(t) \) is constructed from three simpler LTI subsystems, as shown below, with respective impulse responses \( g_1(t) \), \( g_2(t) \), and \( g_3(t) \).

\[ \begin{array}{c}
\text{x(t)} \to \quad g_1(t) \quad \to \quad g_2(t) \quad \to \quad g_3(t) \quad \to \quad y(t) \quad \to \quad h(t)
\end{array} \]

- **a.** (5 points) Find \( h(t) \) in terms of \( g_1(t) \), \( g_2(t) \), and \( g_3(t) \).
- **b.** (5 points) If \( g_1(t) = \delta(t - 2) \), \( g_2(t) = 6\delta(t - 3) \), and \( g_3(t) = -2\delta(t - 1) \), find \( h(t) \).
- **c.** (10 points) Suppose that \( h(t) = \delta(t + 1) + 2\delta(t - 3) \). Find, and accurately sketch \( y(t) = x(t) * h(t) \) for \( x(t) = [u(t + 2) - u(t - 2)] \).
- **d.** (10 points) Suppose that \( h(t) = 2e^{-2t}u(t) \). Find, and sketch, \( y(t) = x(t) * h(t) \) for the same \( x(t) \) as in part c), \( x(t) = [u(t + 2) - u(t - 2)] \).

\[ a) \quad h(t) = g_1(t) * (g_2(t) + g_3(t)) = g_1(t) * g_2(t) + g_1(t) * g_3(t) \]
\[ b) \quad h(t) = 6\delta(t - 5) + \delta(t - 3) \]
\[ c) \quad h(t) = \delta(t + 1) + 2\delta(t - 3) + [x(t) = [u(t + 2) - u(t - 2)]] = y(t)
\]
\[ y(t) = [u(t + 3) - u(t - 1)] + 2[u(t - 1) - u(t - 5)] \]
\[ d) \quad h(t) = 2e^{-2t}
\]
\[ y(t) = 2e^{-2t} \]
\[ y(t) = 0 \text{ for } t < -2 \]
\[ \text{for } -2 \leq t \leq 2 \quad y(t) = \int_{-2}^{t} 2e^{-2t} \, dt = -e \]
\[ \text{for } t \geq 2 \quad y(t) = \int_{-2}^{2} 2e^{-2t} \, dt = e - e \]
4. (25 points) A periodic signal, \( x(t) \), is shown below.

\[
\begin{array}{cccccccccc}
-6 & -4 & -3 & -1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\text{........} & \text{20} & \text{x}(t) & \text{........} \\
\end{array}
\]

\( t, \text{ sec} \)

\[
\sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T}
\]

a. Find the exponential Fourier series, \( x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T} \)

i. (2 points) Period \( T = 5 \)

ii. (3 points) \( c_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt = \frac{20}{5} = 4 \)

iii. (5 points) \( c_k = \frac{20}{5} \left( \frac{\sin \left( \frac{\pi k}{5} \right)}{\frac{\pi k}{5}} \right) \)

\[
c_k = \frac{20}{5} \left( \frac{\sin \left( \frac{\pi k}{5} \right)}{\frac{\pi k}{5}} \right) = 4 \frac{\pi k}{5} \frac{\sin \left( \frac{\pi k}{5} \right)}{\pi k/5}
\]

b. (5 points) Let \( z(t) = 3x(t) + 6 \). Find the exponential Fourier series for \( z(t) \) in terms of the Fourier series coefficients for \( x(t) \).

\[
z(t) = \sum_{k=-\infty}^{\infty} d_k e^{j2\pi kt/T}
\]

\[
d_0 = 3c_0 + 6
\]

\[
d_k = 3c_k, \quad k \neq 0
\]

c. The signal \( x(t) \) in part a) is applied as the input to a linear, time-invariant system with output \( y(t) \) and transfer function \( H(s) = \frac{0.2}{s + 0.4} \).

i. (5 points) Find the Fourier series for \( y(t) \).

ii. (5 points) Find the ratio of the amplitude of the 1st harmonic of the Fourier series for \( y(t) \) to the amplitude of the 1st harmonic of the Fourier series for \( x(t) \).

\[
y(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T}
\]

\[
d_k = c_k \left( \frac{0.2}{j0.4 + D} \right)
\]

\[
\text{ratio} = \frac{|12d_k|}{12c_k} = \frac{|12 \cdot \frac{0.2}{j0.4 + D}|}{12} = \frac{1}{2} \times 0.3535...
\]

\[
k = 1
\]