Signal Energy: \[ ||x(n)||^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2 \]

\( l_2 \) is the set of finite-energy (discrete-time) signals.

Inner Product: \( (x(n), y(n)) = \sum_{n=-\infty}^{\infty} x(n)y^*(n) \)

Note: 1) \( (x(n), x(n)) = ||x(n)||^2 \)

2) Two signals are said to be orthogonal if their inner product is zero, i.e.,

\( (x(n), y(n)) = 0 \iff x(n) \) orthogonal to \( y(n) \)

Example: Find the energy in the signal \( x(n) \) below. Which signals, if any, are orthogonal to \( x(n) \)?

\[ x(n) = (\ldots, 0, 2, 2, -2, -2, 2, 0, \ldots) \]

\[ y_1(n) = (\ldots, 0, -1, -1, 1, 1, -1, -1, 0, \ldots) \]
\[ y_2(n) = (\ldots, 0, 1, 1, 1, 1, -1, 0, \ldots) \]
\[ y_3(n) = (\ldots, 0, -1, -1, -1, 1, 1, 1, 0, \ldots) \]
\[ y_4(n) = (\ldots, 0, 1, 2, 3, 3, 2, 1, 0, \ldots) \]
The cross-correlation of the signals $x(n)$ and $y(n)$ is defined as

$$r_{xy}(k) = (x(n + k), y(n)) = \sum_{n=-\infty}^{\infty} x(n + k)y^*(n)$$

The cross-correlation can be thought of as the inner product between two signals for all offsets (time shifts) of one signal relative to the other.

Using change of variables, it follows that

$$r_{xy}(k) = \sum_{n=-\infty}^{\infty} x(n)y^*(n - k)$$

And

$$r_{yx}(k) = r_{xy}^*(-k)$$

If $y(n) = x(n)$ then we have the autocorrelation function

$$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n + k)x^*(n)$$

and $r_{xx}(k) = r_{xx}^*(-k)$. Hence, the autocorrelation is an even function for real-valued signals.
Properties of Autocorrelation and Cross-correlation

1. Energy in $x(n)$: $r_{xx}(0) = ||x(n)||^2 = E_x$
2. Peak of magnitude: $|r_{xx}(k)| \leq |r_{xx}(0)|$
3. Magnitude is even: $|r_{xx}(k)| = |r_{xx}(-k)|$
4. Normalized autocorrelation: $\rho_{xx}(k) = \frac{r_{xx}(k)}{r_{xx}(0)}$
5. Cross-correlation magnitude

$$|r_{xy}(k)| \leq \sqrt{r_{xx}(0)r_{yy}(0)} = \sqrt{E_xE_y}$$
Example. Find the autocorrelation function of $x(n) = (\cdots, 0, 1, 1, 1, 0, \cdots)$. $r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n + k)x^*(n)$
Suppose that $y(n) = \begin{cases} 1, & 6 \leq n \leq 8 \\ 0, & \text{otherwise} \end{cases}$. Find $r_{xy}(k)$, the cross-correlation of $x(n)$ and $y(n)$. Sketch $x(n)$, $y(n)$ and $r_{xy}(k)$. Observe that $y(n) = x(n - n_0)$. Relate $n_0$ to $r_{xy}(k)$. 

![Graphs of x(n), y(n), r_{xy}(k)]
Example 1 – Determine signal delay.

Suppose we have two signals, \( x(n) \) and \( y(n) \), that we wish to compare. Specifically, let \( x(n) \) be a known reference signal and consider the case where \( y(n) \) is an attenuated and delayed version of \( x(n) \), observed in the presence of noise, and modeled as

\[
y(n) = \alpha x(n - n_0) + w(n)
\]

where \( w(n) \) is a sequence of noise samples, \( \alpha \) is the attenuation factor, and \( n_0 \) is the delay (in samples). The problem is to estimate the delay, \( n_0 \), from an observed signal \( y(n) \).
The figure below shows $x(n)$, $y(n)$, and $r_{xy}(k)$, for zero noise ($w(n) = 0$ for all $n$), $\alpha = 0.1$, and $x(n)$ a 10 ms sinusoid of frequency 1,000 Hz over the duration $[1, 11]$ ms. The sampling rate is $\frac{1}{T} = 20$ kHz.

$$r_{xy}(k) = \sum_{n=-\infty}^{\infty} x(n + k)y^*(n)$$

The peak magnitude of $r_{xy}(k)$ occurs at $k = -720$. The delay is then 720 samples, or

$$|k| \times T = 720 \times \frac{1}{20,000} = 36 \text{ ms}.$$
The figure below repeats the example, but this time the noise is additive white Gaussian noise of zero mean and standard deviation $\sigma = 0.1$. Note that the delayed and attenuated version of $x(n)$ is seemingly buried in the noise, so $x(n)$ is not readily apparent in the observed noisy signal $y(n)$. Using the peak of the cross-correlation function magnitude again leads to $k = -720$ and an estimated signal delay of 36 ms. (The Matlab details of the example are summarized in separate notes “Cross-Correlation Example.” $y(n) = ax(n - n_0) + w(n)$)

Figure 2. Input signal, $x(n)$ vs. time (sec); Delayed/attenuated signal, $y(n)$, vs. time (sec); Noisy delayed/attenuated signal $y(n)$ vs. time (sec); Cross-correlation $r_{xn}(k)$ vs. lag index, $k$. Note the difference between the noisy delayed signal and the cross-correlation compared to Figure 1.
Example 2 – Estimate the pitch period in speech

A speech sentence is bandlimited to 3.5 kHz and sampled at 8 kHz. The first 2,500 samples of the sentence are shown in the top of Figure 3. A segment of 160 samples from the middle of the waveform at the top is shown in the bottom of the figure. Note the “quasi-periodic” nature of the signal.

Problem: Find the fundamental period (pitch period) of the segment shown.

Figure 3. Top – 2,500 samples of a speech waveform; Bottom – 160 samples of the waveform.
From the speech segment shown in the bottom of Figure 3, the autocorrelation function is computed as

\[ r_{xx}(k) = \sum_{n=0}^{159} x(n + k)x(n), \]

the speech segment extracted simply by truncating the original waveform (and hence, \( x(n) = 0 \) for \( n < 0 \) or \( n \geq 160 \)). The autocorrelation function is plotted in the middle of Figure 4. The first peak occurs at \( k = \pm 38 \), so the pitch period is \( T_p = 38 \times T = 4.75 \) ms. The pitch frequency is \( \frac{1}{T_p} = 210.5 \) Hz. The bottom plot in Figure 4 shows the DFT magnitude of the speech segment, the first peak consistent with a pitch frequency of 210.5 Hz.

**Figure 4.** Top – speech segment (same at bottom of Fig. 3); Middle – normalized autocorrelation function for speech segment; Bottom – 256-pt discrete Fourier transform of speech segment data.