And Difference Equation Systems

The basic system model is shown below with input $x(t)$ and output $y(t)$.

\[
\begin{array}{c}
x(n) \\
\rightarrow
\end{array} \quad T[x(n)] \quad \rightarrow \quad y(n)
\]

Observe that the input can be expressed as the identity

\[
x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n - k).
\]

Applying the input to the system, the output is

\[
y(n) = T[x(n)] = T\left[ \sum_{k=-\infty}^{\infty} x(k) \delta(n - k) \right].
\]

Now, suppose that the system is linear. Then

\[
y(n) = T\left[ \sum_{k=-\infty}^{\infty} x(k) \delta(n - k) \right] = \sum_{k=-\infty}^{\infty} T[x(k) \delta(n - k)].
\]

Again, using linearity,

\[
y(n) = \sum_{k=-\infty}^{\infty} x(k) T[\delta(n - k)].
\]
Now, define the time-varying impulse response as

\[ h(n, k) = T[\delta(n - k)]. \]

This is the response at time \( n \) to an impulse (Kronecker) applied at time \( n - k \).

If the system is also time-invariant (shift-invariant) then

\[ h(n, k) = h(n - k), \]

and the input-output equation becomes discrete convolution

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n - k) = \sum_{k=-\infty}^{\infty} h(k) x(n - k). \]

Ex 1. Use discrete convolution to find \( y(n) = x(n) * h(n) \) for \( x(n) = \ldots 0, 1, 2, 2, -1, 1, 0, \ldots \) and \( h(n) = \ldots 0, 2, 1, 1, 0, \ldots \)
Ex. 2. Use discrete convolution to find \( y(n) = x(n) * h(n) \) for \( x(n) = a^n u(n) \) and i) \( h(n) = u(n) \), ii) \( h(n) = b^n u(n) \).
Ex. 3. The “N-tap averager” filter has impulse response
\[ h(n) = \frac{1}{N}(u(n) - u(n - N)). \]

For \( N = 3 \), find
\[ y(n) = x(n) * h(n) \text{ for } x(n) = \cdots, 0, 1, 2, 3, 4, 3, 2, 1, 0, \cdots. \]

Application - \( N = 50 \) and 50-day moving average of S&P.
If an LTI system has impulse response $h(n)$ of finite length, say, $N$, then the system is called finite impulse response (FIR). If the length of $h(n)$ is infinite, then the filter is an infinite impulse response (IIR) filter.

An $N$-tap FIR filter with impulse response

\[ h(n) = h(0), \ldots, h(N - 1) \]

can be implemented as follows (called Direct Form I):

What is the implementation complexity of the FIR filter?
Difference Equation Filters – a broad class of linear, time-invariant systems is defined by the difference equation

\[ y(n) = \sum_{k=1}^{N} a_k y(n - k) + \sum_{k=0}^{M} b_k x(n - k) \]  

(1)

If any of the \( a_k \) are non-zero, then the filter is IIR; if all of the \( a_k \) are zero, then the difference equation in (1) simplifies to

\[ y(n) = \sum_{k=0}^{M} b_k x(n - k) \]

which is the equation for convolution with \( h(k) = b_k \).

A (Direct Form I) realization of the filter in (1) is as follows.
Ex. 5. Find the system impulse response for

\[ y(n) = 0.9y(n-1) + x(n) \]

What is the filter response to the input \( x(n) = u(n) \)?
Two simple FIR filters

Lowpass \( h_{LP}(n) = \frac{1}{2} (1, 1) \) \hspace{1cm} Highpass \( h_{HP}(n) = \frac{1}{2} (1, -1) \)

Step Response: \( x_1(n) = u(n) - u(n - 9) \),

\[
\begin{align*}
y_{LP}(n) &= x(n) * h_{LP}(n) = 0.5(x(n) + x(n - 1)) \\
y_{HP}(n) &= x(n) * h_{HP}(n) = 0.5(x(n) - x(n - 1))
\end{align*}
\]

Response to \( x_2(n) = (0, 1, 2, 3, 4, 3, 2, 1, 0, \ldots) \)