Some types of signal and Fourier analysis

a) continuous-time (analog)

i) periodic (infinite energy)  

\[ x(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j2\pi nt}{T}} \]

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \]

\[ c_n = \frac{1}{T} \int_{0}^{T} x(t) e^{-\frac{j2\pi nt}{T}} \, dt \]

\[ x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df \]

ii) aperiodic (finite energy)

b) discrete-time

iii) periodic  

\[ x(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j2\pi nt}{T}} \]

\[ X(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f-nf_0) \]

\[ c_n = \frac{1}{T} \int_{0}^{T} x(t) e^{-\frac{j2\pi nt}{T}} \, dt \]

\[ x(t) = \sum_{n=-\infty}^{\infty} X(nf_0) e^{j2\pi nt} \]

iv) aperiodic (finite energy)

discrete-time Fourier series  

discrete-time Fourier transform
Discrete-Time Fourier Transform (DTFT)

\[ X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fnT} \quad (= X(e^{j2\pi fT})) \]

Here, \( f \) is frequency in Hertz, and the DTFT is periodic in \( f \) with period \( \frac{1}{T} \).

**Note:** In lecture, homework, notes, tests, etc., the notation \( X(e^{j2\pi fT}) \) will usually be used (to clearly distinguish between the Fourier transform of an analog signal and the DTFT of a discrete-time signal).

The inverse DTFT is given by

\[
x(n) = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(e^{j2\pi fT})e^{j2\pi fn} df
\]

(Set \( T = 1 \) in the expression if using normalized frequency.)
Four expressions for DTFT – all equivalent, but using different frequency variables.

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}, \quad (1) \]

\( \omega \) is in units of normalized frequency, radians/sample, and \( X(e^{j\omega}) \) is periodic \( 2\pi \) in \( \omega \).

\[ X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega nT}, \quad (2) \]

\( \omega \) in units of frequency, radians/sec, and \( X(e^{j\omega T}) \) is periodic \( \frac{2\pi}{T} \) in \( \omega \).

\[ X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn}, \quad (3) \]

\( f \) is in units of normalized frequency, cycles/sample, and \( X(e^{j2\pi f}) \) is periodic 1 in \( f \).

\[ X(e^{j2\pi fT}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fnT}, \quad (4) \]

\( f \) is in units of frequency, cycles/sec (Hertz), and \( X(e^{j2\pi fT}) \) is periodic \( \frac{1}{T} \) in \( f \).
Example 1. Find the DTFT of $x(n) = \alpha^n u(n)$, $|\alpha| < 1$. Plot $|X(e^{j2\pi f T})|$ for real-valued $0 < \alpha < 1$. Modify to units of rad/s.

Relate to the Fourier transform of an analog signal $x_a(t) = e^{-\beta t} u(t)$, sampled at the rate $\frac{1}{T}$ samples/sec. Plot $|X_a(f)|$ and relate to $|X(e^{j2\pi f T})|$. 
Specific case: \( x_a(t) = e^{-2t}u(t) \rightarrow X_a(f) = \frac{1}{2+j2\pi f} \).

Select the sampling rate as \( \frac{1}{T} = 8 \) Hz. Then \( e^{-2T} = \alpha \approx 0.78 \).

The DTFT of \( x(n) = x_a(nT) = e^{-2nT}u(nT) = \alpha^n u(n) \) is \( X(e^{j2\pi fT}) = \frac{1}{1-\alpha e^{-j2\pi fT}} \) with

\[
|X(e^{j2\pi fT})| = \frac{1}{\sqrt{1-2\alpha \cos(2\pi fT)+\alpha^2}}.
\]

Note that \( |X(e^{j2\pi fT})| \) is periodic with period \( \frac{1}{T} = 8 \) Hz. The difference between the two curves is due to aliasing. Note also the scaling of \( |X_a(f)| \) by \( \frac{1}{T} \) in the figure.
Low Frequency and High Frequency

Since the DTFT is periodic in frequency, normally low frequency is zero, and high frequency is one-half the period (that is, one-half the sampling frequency).

Example. Assume a normalized frequency (cycles/sample, or equivalently, $T = 1$). For $x(n) = \alpha^n u(n)$, $|\alpha| < 1$, the DTFT was found to be $X(e^{j2\pi f}) = \frac{1}{1-\alpha e^{-j2\pi f}}$ for the normalized frequency variable $f$ in units of cycles/sample. Then high frequency is $\frac{1}{2}$ cycles/sample.

Alternatively, using the frequency variable $f$ in units of Hz, then $X(e^{j2\pi fT}) = \frac{1}{1-\alpha e^{-j2\pi fT}}$, and high frequency is $\frac{1}{2T}$ Hz.
Existence of the DTFT.

Since the DTFT is a special case of the Z-transform, where z is on the unit circle, the DTFT exists whenever the Z-transform ROC contains the unit circle.

All BIBO stable LTI systems have ROC containing the unit circle, implying the DTFT of the impulse response must exist. \( H(e^{j2\pi fT}) = \text{DTFT}\{h(n)\} \) is the frequency response of a stable LTI system.
Example. A LTI system has impulse response

\[ h(n) = \left( \frac{1}{2}, \frac{1}{2} \right). \]

Determine the transfer function, \( H(z) \), and the DTFT \( H(e^{j2\pi T}) \). Plot \( |H(e^{j2\pi f T})| \). Relate the “dc” gain and the high-frequency gain to the impulse response and to \( H(z) \). Repeat for a LTI system with impulse response \( g(n) = \left( \frac{1}{2}, \frac{-1}{2} \right) \).
Useful Facts and Properties

Let $x(n)$ have DTFT $X(e^{j2\pi fT}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fnT}$.

Write in magnitude/phase form:

$$X(e^{j2\pi fT}) = |X(e^{j2\pi fT})| e^{j\theta(f)}$$

where $\theta(f) = \tan^{-1}\left[\frac{\text{Im}\{X(e^{j2\pi fT})\}}{\text{Re}\{X(e^{j2\pi fT})\}}\right]$.

1. The only causal signal with $\theta(f) = 0$ for all $f$ (that is, zero phase) is $x(n) = A\delta(n)$. Hence, the only zero-phase causal filter is the simple scaling filter. That is, $h(n) = A\delta(n)$, with a system input-output description is $y(n) = Ax(n)$. This is also a (trivial) all-pass filter.
2. A (non-causal) real-valued signal can have zero phase if \( x(n) = x(-n) \) for all \( n \).

Proof. Write out \( X(e^{j2\pi fT}) \):

\[
X(e^{j2\pi fT}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fnT}
\]

\[
= x(0) + x(1)e^{-j2\pi f1T} + x(-1)e^{-j2\pi f(-1)T}
\]

\[
+ x(2)e^{-j2\pi f2T} + x(-2)e^{-j2\pi f(-2)T} + \ldots
\]

\[
X(e^{j2\pi fT})
\]

\[
= x(0) + x(1)[e^{-j2\pi fT} + e^{j2\pi fT}]
\]

\[
+ x(2)[e^{-j2\pi f2T} + e^{j2\pi f2T}] + \ldots
\]

\[
X(e^{j2\pi fT})
\]

\[
= x(0) + x(1)[2 \cos(2\pi fT)]
\]

\[
+ x(2)[2 \cos(4\pi fT)] + \ldots
\]

\[
= \text{real-valued function of } f
\]

Hence, the phase of \( X(e^{j2\pi fT}) \) is

\[
\theta(f) = \tan^{-1}\left[ \frac{\text{Im}\{X(e^{j2\pi fT})\}}{\text{Re}\{X(e^{j2\pi fT})\}} \right]
\]

\[
= \tan^{-1}\left[ \frac{0}{\text{Re}\{X(e^{j2\pi fT})\}} \right] = 0,
\]
provided $\text{Re}\{X(e^{j2\pi fT})\} > 0$.

So, a symmetric signal, $x(n) = x(-n)$ implies zero phase.

Example. The signal $h(n) = (\ldots, 0, -1, 2, 6, 2, -1, 0, \ldots)$ has zero phase.

The DTFT is evaluated as

$$H(e^{j2\pi fT}) = 6 + 4 \cos(2\pi fT) - 2 \cos(4\pi fT)$$

Setting $T = 1$ so frequency is normalized to cycles/sample, the DTFT is plotted below.
Note: The period is 1, and high frequency is $f = \frac{1}{2}$ cycles/sample. The phase is zero for all $f$.

3. DTFT Shifting Property: If $x(n) \xrightarrow{DTFT} X(e^{j2\pi f T})$, then

$$x(n - n_0) \xrightarrow{DTFT} e^{-j2\pi f T n_0} X(e^{j2\pi f T})$$

Proof: Use the shifting property of the Z-transform with $z = e^{j2\pi f T}$.

Linear phase is of the form $\theta(f) = -Af$, for some constant $A$, so a linear phase DTFT has the form

$$H(e^{j2\pi f T}) = |H(e^{j2\pi f T})| e^{-jAf}.$$

4. A causal FIR filter with impulse response $h(0), \ldots, h(N - 1)$ has linear phase if

$$h(n) = h(N - 1 - n), \quad n = 0, 1, \ldots, N - 1.$$  

Proof. The DTFT is

$$H(e^{j2\pi f T}) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi fnT}$$

$$= h(0) + h(1)e^{-j2\pi f T} + h(2)e^{-j2\pi f 2T} + \ldots$$

$$+ h(N - 2)e^{-j2\pi f (N-2)T}$$

$$+ h(N - 1)e^{-j2\pi f (N-1)T}$$
\[ H(e^{j2\pi fT}) \]
\[ = h(0)\left[ 1 + e^{-j2\pi f(N-1)T} \right] \]
\[ + h(1)\left[ e^{-j2\pi f1T} + e^{-j2\pi f(N-2)T} \right] + \ldots \]

Now, factor out the same factor from each complex exponential, namely \( e^{\frac{j2\pi f(N-1)T}{2}} \). Then

\[ H(e^{j2\pi fT}) \]
\[ = e^{\frac{-j2\pi f(N-1)T}{2}} \left( h(0) \left[ e^{\frac{j2\pi f(N-1)T}{2}} \right. \right. \]
\[ + e^{\frac{-j2\pi f(N-1)T}{2}} \]
\[ + h(1) \left[ e^{\frac{j2\pi f(N-3)T}{2}} + e^{\frac{-j2\pi f(N-3)T}{2}} \right] + \ldots \right) \]

\[ H(e^{j2\pi fT}) \]
\[ = e^{\frac{-j2\pi f(N-1)T}{2}} \left( h(0) \cos \left[ \frac{2\pi f(N-1)T}{2} \right] \right. \]
\[ + h(1) \cos \left[ \frac{2\pi f(N-3)T}{2} \right] + \ldots \right) \]

Hence, the phase of \( H(e^{j2\pi fT}) \) is
\[ \theta(f) = \frac{-2\pi f(N - 1)T}{2}. \]
Example. A LTI system has impulse response

\[ h(n) = \frac{1}{8}(-1, 2, 6, 2, -1) \]. The length is \( N = 5 \). By inspection, \( h(n) = h(N - 1 - n) \), \( n = 0, 1, 2, 3, 4 \), so this is a linear-phase filter. What is the “dc” gain? What is the high-frequency gain? What type of filter is this?

Example. A LTI system has impulse response

\[ h(n) = \frac{1}{4}(1, 2, 1) \]. Find the DTFT \( H(e^{j2\pi f T}) \). Use convolution to find the response to input

\[ x(n) = (\cdots 0, 1, 2, 3, 4, 3, 2, 1, 0 \cdots) \]. Relate the filtering time delay to the DTFT.
5. Convolution property.

\[ y(n) = x(n) * h(n) \]

\[ \DTFT X(e^{j2\pi fT})H(e^{j2\pi fT}) \quad (1) \]

and

\[ y(n) = x(n)h(n) \DTFT X(e^{j2\pi fT}) \]

\[ * H(e^{j2\pi fT}) \quad (2) \]

where convolution in the frequency domain is periodic convolution defined as

\[ X(e^{j2\pi fT}) * H(e^{j2\pi fT}) = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(e^{j2\pi(f-\lambda)T})H(e^{j2\pi \lambda T})d\lambda \]

Proof of (1) follows directly from the convolution property of the Z-transform, with \( z = e^{j2\pi fT} \).

Proof of (2). Begin with the right-hand side of (2) and take the inverse DTFT.
6. Parseval’s Relation

\[ E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |X(e^{j2\pi fT})|^2 df \]

Proof.
7. Let \( x(n) \stackrel{DTFT}{\longrightarrow} X(e^{j2\pi f T}) \). In general, \( x(n) \) and \( X(e^{j2\pi f T}) \) are complex-valued. Write the real and imaginary parts as the signal name with subscript \( R \) or \( I \). So

\[
x(n) = x_R(n) + jx_I(n) \\
X(e^{j2\pi f T}) = X_R(e^{j2\pi f T}) + jX_I(e^{j2\pi f T})
\]

a. If \( x(n) \) real-valued then \( X_R(e^{j2\pi f T}) \) is even in frequency \( f \) and \( X_I(e^{j2\pi f T}) \) is odd in frequency, so the magnitude, \(|X(e^{j2\pi f T})|\) is even, and the phase, \( \angle X(e^{j2\pi f T}) \) is odd in frequency.

b. If \( x(n) \) real-valued and even then \( X(e^{j2\pi f T}) \) is real and even.

c. If \( x(n) \) real-valued and odd then \( X(e^{j2\pi f T}) \) is imaginary and \( X_I(e^{j2\pi f T}) \) is odd.

d. If \( x(n) \) imaginary then \( X_R(e^{j2\pi f T}) \) is odd and \( X_I(e^{j2\pi f T}) \) is even.

8. Time reversal. If \( x(n) \stackrel{DTFT}{\longrightarrow} X(e^{j2\pi f T}) \) then

\[
x(-n) \stackrel{DTFT}{\longrightarrow} X(e^{-j2\pi f T})
\]

Proof. This property follows directly from the \( Z \)-transform property.
9. Energy Density Spectrum and Autocorrelation

Let $x(n)$ be an energy signal with $x(n) \overset{DTFT}{\longrightarrow} X(e^{j2\pi fT})$. The magnitude squared of the DTFT is referred to as the energy density spectrum, $S_{xx}(e^{j2\pi fT}) = |X(e^{j2\pi fT})|^2$. From Parseval’s relation, the signal energy is

$$
E_x = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} S_{xx}(e^{j2\pi fT}) df.
$$

Property: The energy density spectrum is related to the autocorrelation as $r_{xx}(k) \overset{DTFT}{\longrightarrow} S_{xx}(e^{j2\pi fT})$, where the autocorrelation function is $r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n + k)x^*(n)$. Proof. Directly compute the DTFT of $r_{xx}(k)$. 

Relate LTI System Transfer Function Poles and Zeros to Filter Frequency Response

Suppose we have a causal LTI system described by

\[ y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k). \]

with transfer function \[ H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}. \] The system has poles and zeros, and the numerator and denominator can be factored into the form

\[ H(z) = \frac{B(z)}{A(z)} = \frac{K \prod_{i=1}^{M} (1 - r_i z^{-1})}{\prod_{i=1}^{N} (1 - p_i z^{-1})} \]

where we assume \( M \) zeros at locations \( z = r_i \) and \( N \) poles at locations \( z = p_i \) and no pole-zero cancellations. Simple manipulation yields

\[ H(z) = \frac{K z^{N-M} \prod_{i=1}^{M} (z - r_i)}{\prod_{i=1}^{N} (z - p_i)}. \]

The magnitude frequency response is obtained by setting \( z = e^{j2\pi f T} \) and evaluating the magnitude of \( H(e^{j2\pi f T}) \) as

\[ |H(e^{j2\pi f T})| = \frac{|K||e^{j2\pi f T(N-M)}| \prod_{i=1}^{M} |(e^{j2\pi f T} - r_i)|}{\prod_{i=1}^{N} |(e^{j2\pi f T} - p_i)|} \]
In the $z$-plane, this is

$$|H(e^{j2\pi f_T})| = \frac{|K| \prod_{i=1}^M \text{dist from } e^{j2\pi f_T} \text{ to zero } r_i}{\prod_{i=1}^N \text{dist from } e^{j2\pi f_T} \text{ to pole } p_i}$$

Now, what about phase? The frequency response can be written as (assuming $K$ is positive)

$$H(e^{j2\pi f_T}) = \frac{|K| \prod_{i=1}^M |e^{j2\pi f_T} - r_i| e^{j\varphi_i}}{\prod_{i=1}^N |e^{j2\pi f_T} - p_i| e^{j\theta_i}}$$

where $\varphi_i = \angle(e^{j2\pi f_T} - r_i)$ is the angle to the zero and $\theta_i = \angle(e^{j2\pi f_T} - p_i)$ is the angle to the pole. Combining terms

$$\angle H(e^{j2\pi f_T}) = \frac{|K| \prod_{i=1}^M |e^{j2\pi f_T} - r_i| e^{j[\sum_{i=1}^M \varphi_i - \sum_{i=1}^N \theta_i]}}{\prod_{i=1}^N |e^{j2\pi f_T} - p_i|}.$$ 

So, the phase of $H(e^{j2\pi f_T})$ equals the sum of the phases from the zeros, minus the sum of the phases from the poles.
Example. Let $H(z) = \frac{1}{1 - 0.9z^{-1}} = \frac{z}{z - 0.9}$.

Sketch the pole-zero locations in the $z$-plane, and then the magnitude and phase frequency response.
Example 2. Let $H(z) = \frac{2(1-z^{-1}0.9)}{(1+0.81z^{-2})}$. Sketch the magnitude frequency response of the filter.

First, sketch the pole and zero locations in the $z$-plane.
Comments about frequency response

Begin at $f = 0$ and traverse the unit circle as $e^{j2\pi fT}$.

- As $e^{j2\pi fT}$ approaches a zero, one term in the numerator of the frequency response decreases rapidly, so $|H(e^{j2\pi fT})|$ decreases. As $e^{j2\pi fT}$ moves away from a zero, then $|H(e^{j2\pi fT})|$ increases.
- As $e^{j2\pi fT}$ approaches a pole, one term in the denominator of the frequency response decreases rapidly, so $|H(e^{j2\pi fT})|$ increases.
- A zero exactly on the unit circle causes the gain to go to zero.
- For poles or zeros near the unit circle, the phase suddenly changes as $e^{j2\pi fT}$ sweeps past.
- Poles or zeros at the origin do not affect the magnitude frequency response.
10. Time shifting and frequency shifting. If 
\[ x(n) \xrightarrow{DTFT} X(e^{j2\pi fT}) \], then

a. \[ x(n - n_0) \xrightarrow{DTFT} e^{-j2\pi fTn_0}X(e^{j2\pi fT}) \]

b. \[ e^{j2\pi f_0 nT} x(n) \xrightarrow{DTFT} X(e^{j(2\pi f - 2\pi f_0)T}) \]

Proof. Direct computation of the DTFT and inverse DTFT.

Example. We already know that \( X(e^{j2\pi fT}) \) is periodic in \( f \) with period \( \frac{1}{T} \). So, if \( f_0 = \frac{k}{T} \) for any integer \( k \), then
\[
X(e^{j(2\pi f - 2\pi f_0)T}) = X(e^{j2\pi fT}).
\]

Example. If \( y(n) = (-1)^n x(n) \), find \( Y(e^{j2\pi fT}) \) in terms of \( X(e^{j2\pi fT}) \).

Application: Suppose we have a good lowpass FIR filter with transfer function \( H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} \), where \( h(n) \) is the length-\( N \) impulse response. How can we easily design a length-\( N \) FIR high-pass filter? Use frequency shifting.
Example. For the causal, linear phase lowpass FIR filter with impulse response \( h(n) = \frac{1}{8}(-1, 2, 6, 2, -1) \), the DTFT is evaluated as

\[
H(e^{j2\pi f T}) = \frac{e^{-j2\pi 2T}}{8} \left[ 6 + 4 \cos(2\pi f T) - 2 \cos(4\pi f T) \right]
\]

and is plotted below for \( T = 1 \).

The delay is \( \frac{N-1}{2} = 2 \) samples.
Now, to obtain a high-pass filter via frequency translation, we wish to shift by half a period (that is, by $\frac{1}{2T}$ Hz) in frequency. Let $G(e^{j2\pi fT}) = H \left(e^{j(2\pi(f - \frac{1}{2T}))T}\right) = H((-1)e^{j2\pi fT})$, so (replacing $e^{j2\pi fT} = z$) it follows that $G(z) = H(-z)$, and

$$G(z) = H(-z) = \sum_{n=0}^{N-1} h(n)(-z)^{-n} = \sum_{n=0}^{N-1} h(n)(-1)^n z^{-n}.$$ 

Hence,

$$g(n) = (-1)^n h(n)$$

is the mapping to convert the lowpass filter to a highpass filter. For $h(n) = \frac{1}{8}(-1, 2, 6, 2, -1)$, this implies that $g(n) = \frac{1}{8}(-1, -2, 6, -2, -1)$.

Note the “dc” gain, and the high-frequency gain, of each filter.
Comparing the two frequency responses, the highpass filter derived by frequency translation,

\[ G(e^{j2\pi fT}) = H(e^{j(2\pi(f - \frac{1}{2T})T)} \]

In summary, to convert an FIR lowpass filter with impulse response \( h(n) \) to an FIR highpass filter with impulse response \( g(n) \), using frequency shifting, simply let \( g(n) = (-1)^n h(n) \).
Example. A few causal lowpass and highpass FIR filters are shown below. The highpass filter impulse response, \( g(n) \), is formed from the lowpass filter impulse response, \( h(n) \), using \( g(n) = (-1)^n h(n) \).

Lowpass filter impulse and frequency responses (N=2,3,4,5).

Highpass filter impulse and frequency response (N=2,3,4,5).
Problem: Suppose we have an IIR lowpass filter given by the difference equation

\[ y(n) = \sum_{k=1}^{N} a_k y(n - k) + \sum_{k=0}^{M} b_k x(n - k). \]

How can we use frequency translation to modify the lowpass filter into a highpass filter?
Design Problem. Design a non-causal lowpass filter with sampling rate \( \frac{1}{T} \) Hz and cutoff frequency \( f_c \) Hz.

Solution. The ideal lowpass filter frequency response is

\[
H(e^{j2\pi fT}) = \begin{cases} 
1, & 0 \leq f \leq f_c \\
0, & f_c < f \leq \frac{1}{2T}
\end{cases}
\]

Use the definition of inverse DTFT to compute \( h(n) \).

\[
h_{\text{ideal}}(n) = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} H(e^{j2\pi fT})e^{j2\pi fn} df
\]

\[
= T \int_{-f_c}^{f_c} e^{j2\pi fn} df = 2f_c T \text{sinc}(2f_c Tn)
\]

A truncated non-causal impulse response is defined as the 2\( N \) + 1 samples of \( h(n) \) over the range \( n = -N, ..., 0, ... N \),

\[
h_N(n) = \begin{cases} 
h_{\text{ideal}}(n), & -N \leq n \leq N \\
0, & \text{otherwise}
\end{cases}
\]
The figures below show $h_N(n)$ and $H_N(e^{j2\pi fT})$ for $T = 1$, $f_c = 0.2$, and $N = 4, 20, \text{ and } 50$, corresponding to impulse response lengths of 9, 41, and 101.
Note: The filters above are **non-causal**. They satisfy the symmetry condition \( h_N(n) = h_N(-n) \), so they have zero phase and \( H_N(e^{j2\pi fT}) \) is real-valued.

**Causal** filters with the same frequency response magnitude can be constructed simply by using the DTFT delay property and forming the causal impulse response as

\[
h(n) = h_N(n - N), \quad n = 0, \cdots, 2N,
\]
yielding the length \( M = 2N + 1 \) FIR filter with frequency response

\[
H(e^{j2\pi fT}) = e^{-j2\pi fNT} H_N(e^{j2\pi fT}).
\]

From the construction, the causal filter impulse response satisfies the condition

\[
h(n) = h(M - 1 - n), \quad n = 0, \cdots, M - 1
\]
so the FIR filter has linear phase given by
\[
\theta(f) = -\frac{2\pi f T (M-1)}{2} = -2\pi f TN.
\]
This example then provides a design strategy for designing causal, linear phase, FIR filters.

1. Design a zero phase non-causal (generally infinite length) filter impulse response using the inverse DTFT.

2. Truncate the infinite-length impulse response to the sample range $n = -N, ..., 0, ... N$. Provided $h(n) = h(-n)$, the truncated filter is still zero phase.

3. Delay the impulse response values by $N$, yielding a length $M = 2N - 1$ causal FIR filter.

Note: This general method is known as the window method of linear phase FIR filter design. In general, the truncation window need not be rectangular, but does need to preserve the required symmetry condition, $h(n) = h(M - 1 - n)$, for $n = 0, \ldots, M - 1$, with causal impulse response length $M$. 
Related Design Problem: Design a length-M linear-phase, causal FIR lowpass filter that is optimum in the MSE sense. That is, if the designed filter has frequency response $H(e^{j2\pi fT})$, and the ideal (causal, linear-phase) lowpass filter has the frequency response,

$$H_{\text{ideal}}(e^{j2\pi fT}) = \begin{cases} 1 & 0 \leq f \leq f_c \\ e^{-j2\pi f (M-1)/2} & f_c < f \leq \frac{1}{2T} \end{cases}$$

Then design $H(e^{j2\pi fT})$ to minimize the integral squared error (mean-square error)

$$\frac{1}{2T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |H_{\text{ideal}}(e^{j2\pi fT}) - H(e^{j2\pi fT})|^2 df.$$

Solution. Recall Parseval’s relation:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |X(e^{j2\pi fT})|^2 df.$$
Digital Frequency

Suppose an analog signal, \( x_a(t) \), is bandlimited to \( B \) Hz, meaning that \( X_a(f) = 0 \) for \( |f| \geq B \) Hz.

The Nyquist sampling rate is \( \frac{1}{T} = 2B \) samples/sec.

Let \( x_a(t) = \cos(2\pi f_0 t) \) so that the samples are
\[
x(n) = \cos(2\pi f_0 nT).
\]

a) If \( f_0 = 0 \), then \( x(n) = (\ldots, 1, 1, 1, \ldots) \) (a “dc” signal)

b) If \( f_0 = \frac{1}{4}T \), then
\[
x(n) = \cos(2\pi \frac{1}{4T} nT) = \cos(\frac{\pi}{2} n)
\]
\[
= (\ldots, 1, 0, -1, 0, -1, 0, -1, 0, 1, \ldots)
\]
Hence, there is a sign change every other sample.

c) If \( f_0 = \frac{1}{2}T \), then
\[
x(n) = \cos(2\pi \frac{1}{2T} nT) = \cos(\pi n)
\]
\[
= (\ldots, 1, -1, -1, -1, 1, \ldots)
\]
There is a sign change every sample.
Note the locations of \( f = 0, \frac{1}{4T}, \frac{1}{2T} \) on the unit circle in the z-plane.
Convolution and Steady-State Response

Suppose we are given a causal, stable, LTI system with impulse response $h(n)$. What is the steady-state response to a sinusoidal input, e.g., to input $x(n) = A \sin(2\pi f_0 n)$?

Derivation. Assume a normalized sampling rate, so $\frac{1}{T} = 1$. Let the system input be $x(n) = A e^{j2\pi f_0 n}$ for all $n$. The system output is given by convolution,

$$y(n) = h(n) * x(n).$$

Evaluating,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n - k) = \sum_{k=-\infty}^{\infty} h(k)A e^{j2\pi f_0 (n-k)}$$

$$= A e^{j2\pi f_0 n} \sum_{k=-\infty}^{\infty} h(k)e^{-j2\pi f_0 k}$$

$$= H(e^{j2\pi f_0})A e^{j2\pi f_0 n}$$

Hence, the output is just a scaled version of the input, where the scale factor is the transfer function evaluated at the input frequency. This result also holds for the steady-state
response if the input begins at some finite time index, and the output is observed after a sufficiently large delay.

Example. Let \( h(n) = \frac{1}{4} (1, 2, 1) \). Determine the steady state response to the following signals (assume \( T = 1 \)).

i) \( x(n) = 12 u(n) \)

ii) \( x(n) = 60 \sin \left( \frac{\pi}{2} n \right) u(n) \)

iii) \( x(n) = 8 \cos(\pi n) u(n) \)

Solution.

\[
H(e^{j2\pi f}) = \frac{1}{4} \left( 1 + 2 e^{-j2\pi f} + e^{-j4\pi f} \right) = \frac{1}{2} e^{-j2\pi f} (1 + \cos(2\pi f)).
\]

\( x(n) = 12 u(n) \Rightarrow y_{\text{steady state}}(n) = H(e^{j0})12 = 12. \)

\( x(n) = 60 \sin \left( \frac{\pi}{2} n \right) u(n) \Rightarrow y_{\text{steady state}}(n) = |H(e^{j0.5\pi})|60 \sin \left( \frac{\pi}{2} n + \theta \right), \)

where \( \theta = \text{phase} \left( H(e^{j0.5\pi}) \right) \).

Evaluating, \( H(e^{j0.5\pi}) = \frac{1}{2} e^{-j0.5\pi} (1 + \cos \left( \frac{\pi}{2} \right)) = \frac{-j}{2} \), so
= |H(e^{j0.5\pi})| = \frac{1}{2} \text{ and } \theta = \frac{-\pi}{2}. \text{ Hence, the steady-state response to the input in ii) is } y_{\text{steadystate}}(n) = \frac{1}{2} 60 \sin \left(\frac{\pi}{2} n + \frac{-\pi}{2}\right) = 30 \sin \left(\frac{\pi}{2} (n - 1)\right).

Note the delay of one sample, matching the linear phase term in the expression for \(H(e^{j2\pi f})\).
Inverse Filters, Minimum Phase, and All-Pass Filters

Definition: A LTI system is called minimum phase if all zeros are inside the unit circle. (Hence, a stable minimum phase LTI system is invertible as a causal filter that is also stable.)

An interesting filter:

\[ H(z) = \frac{1 - \frac{1}{\alpha} z^{-1}}{1 - \alpha z^{-1}}. \]

The pole is at \( z = \alpha \) and we assume that \( |\alpha| < 1 \) so the filter is stable as a causal filter. The zero is at \( z = \frac{1}{\alpha} \), and since \( |\alpha| < 1 \), the zero is outside the unit circle. Hence, this filter is non-minimum phase.

Assume that \( \alpha \) is real-valued and examine the frequency response.

\[ H(e^{j2\pi fT}) = \frac{1 - \frac{1}{\alpha} e^{-j2\pi fT}}{1 - \alpha e^{-j2\pi fT}} = \frac{e^{-j2\pi fT}(\alpha e^{j2\pi fT} - 1)}{\alpha(1 - \alpha e^{-j2\pi fT})}. \]

Noting that the two terms in parentheses are related as \( (\alpha e^{j2\pi fT} - 1) = -(1 - \alpha e^{-j2\pi fT})^* \), then \( |H(e^{j2\pi fT})| = \frac{1}{|\alpha|} \), since \( \alpha \) is real-valued.
Hence, \( H(z) = \frac{\frac{1}{\alpha}z^{-1}}{1-\alpha z^{-1}} \) is an **all-pass** filter, with magnitude frequency response \( |H(e^{j2\pi f T})| = \frac{1}{|\alpha|} \), i.e., constant for all \( f \).

Now, what kind of pole-zero configurations are possible in an all-pass filter?

For a difference equation LTI system with real-valued coefficients, all complex-valued poles of zeros must appear in complex-conjugate pairs. So, if \( z = \alpha \) is a pole, so must be \( z = \alpha^* \). For an all-pass filter, if \( z = \alpha \) is a pole, then \( z = \frac{1}{\alpha} \) must be a zero. So, if \( \alpha \) is complex-valued, then \( z = \alpha \) and \( z = \alpha^* \) are poles, implying that \( z = \frac{1}{\alpha} \) and \( z = \frac{1}{\alpha^*} \) must be zeros. Hence, in general, poles and zeros must appear in “quads” in all-pass filters.

If a difference equation all-pass filter is causal and BIBO stable, is the inverse filter BIBO stable as a causal filter?
Notch Filters

A notch filter can be designed to eliminate a specific frequency component, say at frequency $f_0$ Hz, by placing a zero of the transfer function at $z = e^{j2\pi f_0 T}$. Assuming this zero location is complex-valued, then a zero need also be placed at $z = e^{-j2\pi f_0 T}$ so that the filter coefficients are real-valued. Placing poles near the zeros, but inside the unit circle, results in a transfer function of the form

$$H(z) = \frac{(1 - e^{j2\pi f_0 T} z^{-1})(1 - e^{-j2\pi f_0 T} z^{-1})}{(1 - \alpha e^{j2\pi f_0 T} z^{-1})(1 - \alpha e^{-j2\pi f_0 T} z^{-1})},$$

where $0 \leq \alpha < 1$ controls how close the pole is to the zero. Simplifying, this becomes

$$H(z) = \frac{1 - 2 \cos(2\pi f_0 T) z^{-1} + z^{-2}}{1 - 2\alpha \cos(2\pi f_0 T) z^{-1} + \alpha^2 z^{-2}}.$$

Example. Design a notch filter to remove 180 Hz, assuming a sampling rate of 1,000 Hz.

Solution. Use $f_0 = 180$ and $T = 10^{-3}$ in the equation above, yielding a transfer function

$$H(z) = \frac{1 - 2 \cos(0.36\pi) z^{-1} + z^{-2}}{1 - 2\alpha \cos(0.36\pi) z^{-1} + \alpha^2 z^{-2}}.$$
The figure below shows the frequency response magnitude for $\alpha = 0$, $\alpha = 0.9$, and $\alpha = 0.99$.

Note how the width of the notch depends on $\alpha$, and that without poles near the zeros the frequency response is not flat for frequencies significantly distant from $f_0 = 180$ Hz.
Comb Filters

These filters can be used to eliminate a periodically spaced set of frequencies, or simply to have a periodic shaped frequency response. We can generate a comb filter from a prototype filter, say $H(z)$. A comb filter is formed as $H_L(z) = H(z^L)$, for any integer $L > 1$. The comb filter frequency response is then

$$H_L(e^{j2\pi fT}) = H(e^{j2\pi fTL}).$$

Since $H(e^{j2\pi fT})$ is periodic in frequency with period $\frac{1}{T}$, then as $f$ varies over the range $\left[0, \frac{1}{T}\right]$ Hz, $H(e^{j2\pi fTL})$ is varying over $L$ periods. Hence, $H_L(e^{j2\pi fT})$ looks like $L$ periods of $H(e^{j2\pi fT})$, but compressed into the frequency range $\left[0, \frac{1}{T}\right]$ Hz.

Example 1. Select $L = 3$ and the prototype filter that has impulse response $h_0(n) = (0.5, 0.5)$. Then

$$H(z) = \frac{1}{2} (1 + z^{-1}).$$

The comb filter has transfer function $H_L(z) = \frac{1}{2} (1 + z^{-L})$, so that with $L = 3$, $H_3(z) = \frac{1}{2} (1 + z^{-3})$, implying an impulse
response $h_0(n) = (0.5, 0, 0, 0.5)$. The frequency responses are shown in the figure below.

Example 2. A simple IIR filter has transfer function

$$H(z) = \frac{1}{1 - 0.9z^{-1}}.$$  

If $L = 4$, the comb filter has transfer function

$$H_L(z) = \frac{1}{1 - 0.9z^{-L}}.$$  

The frequency responses are shown below.
Question: Suppose we have a difference equation LTI digital filter. How can we implement the comb filter?

Answer: Just replace every delay, $z^{-1}$, with $z^{-L}$ in the realization.