The process of up-sampling by integral factor $L$, also referred to as zero-insertion (or somewhat ambiguously as interpolation) is diagrammed as shown

$$y(n) = \begin{cases} x \left( \frac{n}{L} \right), & n = rL, r \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

for integer $L \geq 1$. That is, there are $L - 1$ zeros placed between each sample of $x(n)$ to produce the signal $y(n)$. If the sample rate for the signal $x(n)$ is $1/T$ samples/sec, then the sample rate for the signal $y(n)$ is $L/T$ samples/sec.

Find the $z$-transform of $y(n)$.

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n} =$$
Let $1/T$ be the sampling rate for $x(n)$ and $\frac{1}{\tau} = \frac{L}{T}$ the sampling rate for $y(n)$. Find DTFT $Y(e^{j2\pi f \tau})$ in terms of $X(e^{j2\pi f T})$. Sketch the two DTFTs for $L = 2$ or $L = 3$. 

\[ \begin{array}{c}
\text{x(n)} \\
\uparrow L \\
\text{y(n)} \\
\end{array} \]
A down-sampler by integral factor $M$ is diagrammed below

\[ \begin{array}{c}
\text{x(n)} \\
\downarrow M \\
\text{y(n)}
\end{array} \]

and defined as $y(n) = x(nM)$.

So, if the sampling rate for the signal $x(n)$ is $1/T$ samples/sec, then the data rate for $y(n)$ is $\frac{1}{\tau} = \frac{1}{MT}$.

Find the $z$-transform of $y(n)$, expressed in terms of the $z$-transform of $x(n)$.

\[
Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(nM)z^{-n}
\]

\[
= \sum_{r=0, \pm M, \pm 2M, \ldots}^{\infty} x(r)z^{-r/M}
\]
Given the down-sampling system below, assume an example spectrum for $X(e^{j2\pi f_T})$, and sketch the spectrum for $Y(e^{j2\pi f_T})$. 

![Diagram of down-sampling system]
Sample Rate Conversion by rational factor $L/M$

Suppose that we have a signal $x(n)$ with sample rate $F_1 = \frac{1}{T}$ samples/sec and we wish to convert to a signal, $y(n)$, with sampling rate $F_2 = \frac{1}{\tau}$, where $F_2 = \frac{L}{M} F_1$, with $L$ and $M$ positive integers. Define frequency

$$f_c = \min\left\{\frac{1}{2T}, \frac{L}{M2T}\right\}.$$  

The following block diagram describes the processing necessary to perform the sample rate conversion, where $f_c$ is the cutoff frequency of the lowpass filter, $H(z)$, in the diagram.

Note: The Matlab resample( ) function does the process automatically.

Draw spectra of analog signal, $x_a(t)$, signal sampled at $1/T$ Hz, $x(n)$, and signal sampled at $L/MT$ Hz, $y(n)$. 
Example 1. Suppose we have an audio signal that is bandlimited to 4 kHz, and sampled at a rate of 8 kHz. Describe how to convert the signal sampling rate to 12 kHz.

Solution. Selecting $L = 3, M = 2$, then using $F_{s,old} = 8,000$ we have

$$F_{s,new} = \frac{L}{M} F_{s,old} = \frac{3}{2} \times 8,000 = 12,000.$$

The required lowpass filter must have cutoff frequency $f_c = \min\left\{\frac{1}{2T}, \frac{L}{M2T}\right\} = \min\{4 \text{ kHz}, 6 \text{ kHz}\} = 4 \text{ kHz}.$

The block diagram is shown below.

The figure below shows example spectra at the points labeled A, B, C, and D in the figure above.
The figure below shows example spectra at the input to the sampler (point “A”), where the frequency labeled B corresponds to $\frac{1}{2T} = 4,000$ Hz; the output of the up-sampler (point “B”); the lowpass filter frequency response, the output of the lowpass filter (point “C”); and at the output of the down-sampler (point “D”).
Example 2. Suppose we have an audio signal that is bandlimited to 4 kHz, and sampled at a rate of 8 kHz. Describe how to convert the signal sampling rate to 6 kHz.

Solution. In this case the final sampling rate is less than the original, so some filtering of the original signal spectrum is required. Selecting $L=3$, $M=4$, then

$$F_{s,new} = \frac{L}{M} F_{s,old} = \frac{3}{4} \times 8,000 = 6,000.$$ 

The filter cutoff frequency is

$$f_c = \min\left\{ \frac{1}{2T}, \frac{L}{M2T} \right\} = \min\{4 \text{ kHz}, 3 \text{ kHz}\} = 3 \text{ kHz}.$$ 

A block diagram of the required processing is shown below.

The figure below shows example spectra at the points labeled A, B, C, and D in the figure above.
The figure below shows example spectra at the input to the sampler (point “A”), where the frequency labeled B corresponds to $\frac{1}{2T} = 4,000$ Hz; the output of the up-sampler (point “B”); the lowpass filter frequency response with cutoff frequency 3,000 Hz; the output of the lowpass filter (point “C”); and at the output of the down-sampler (point “D”). Note that the information in the original spectrum between 3 kHz and 4 kHz has been filtered away.

$3B/4 = 3,000 \text{ Hz}$