Define the $Z$-transform of the sequence $x(n)$ as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n},$$

valid for all $z \in \mathbb{C}$ for which $X(z)$ converges, where $\mathbb{C}$ is the set of complex numbers. Note that $X(z)$ is just a power series in $z^{-1}$, with the signal samples the series coefficients.

Example 1. Find the $Z$-transform of $x(n) = a^n u(n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \lim_{N \to \infty} \sum_{n=0}^{N} (az^{-1})^n$$

$$X(z) = \lim_{N \to \infty} \left\{ \begin{array}{ll}
N + 1, & \text{for } z = a \\
\frac{1 - (az^{-1})^{N+1}}{1 - az^{-1}}, & \text{for } z \neq a
\end{array} \right.$$  

Taking the limit, get convergence only if $|az^{-1}| < 1$, and the $Z$-transform is $X(z) = \frac{1}{1-az^{-1}}$, ROC $|a| < |z|$.

Summarize as $a^n u(n) \xrightarrow{Z} \frac{1}{1-az^{-1}}$, ROC $|a| < |z|$.
Example 2. Find the Z-transform of the “left-sided signal”

\[ x(n) = \begin{cases} 
-a^n, & n < 0 \\
0, & n \geq 0 
\end{cases} = -a^n u((-1) - n). \]

Summarize as

\[ -a^n u(-1 - n) \xrightarrow{z} \frac{1}{1 - az^{-1}}, \quad \text{ROC } |z| < |a| \]
Example 3. Find the Z-transform of the “double-sided signal” \( x(n) = \begin{cases} a^n, & n \geq 0 \\ -b^n, & n < 0 \end{cases} \)

\[
X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = -\sum_{n=-\infty}^{-1} b^n z^{-1} + \sum_{n=0}^{\infty} a^n z^{-1}
\]

\[
X(z) = \frac{1}{1 - az^{-1}} + \frac{1}{1 - bz^{-1}}, \quad \text{ROC} \quad |a| < |z| < |b|
\]

If \(|a| > |b|\) then the ROC is empty and the transform does not exist.
Aside: Where does the Z-transform come from?
Consider the “impulse sampling” model of the sampling process.

\[ x_\sigma(t) = x_a(t) \sum_{n=0}^{\infty} \delta(t - nT) \]

Now take the Laplace transform (assume \( n \geq 0 \))

\[
X_\sigma(s) = \int_0^\infty x_\sigma(t) e^{-st} dt = \int_0^\infty \sum_{n=0}^{\infty} x_a(t) \delta(t - nT) e^{-st} dt \\
= \sum_{n=0}^{\infty} x_a(nT) e^{-snT}.
\]

So, with the definition \( z = e^{sT} \) this is just the formula for the Z-transform \( X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \), where \( x(n) = x_a(nT) \) is the discrete-time signal.
Mapping from $s$-plane to $z$-plane:

Note: If $Re\{s\} < 0$

If $Re\{s\} = 0$

If $Re\{s\} > 0$

Note also: $z = e^{sT}$ is a many-to-one mapping from the $s$-plane to the $z$-plane.
Observation: The $Z$-transform region of convergence (ROC) is always an annulus (a ring in the $z$-plane) of the form $R_1 < |z| < R_2$.

Proof: Write the $Z$-transform as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{0} x(n)z^{-n} + \sum_{n=0}^{\infty} x(n)z^{-n} - x(0).$$

Now, consider the first term and call it $A(z)$. Using a change of variables, there are only positive powers of $z$

$$A(z) = \sum_{n=0}^{\infty} x(-n)z^{n}.$$

Fact: If $A(z)$ converges for some value $z_1$, then $A(z)$ must converge for every $z$ satisfying $|z| < |z_1|$.

Proof. Suppose convergence for $z = z_1$, so

$$\sum_{n=0}^{\infty} x(-n)z_1^n = A_1$$

for some finite value $A_1$. A necessary condition for any series to converge is that each term in the series is
bounded. Thus, there must be some finite non-negative number, say $M$, satisfying $0 \leq M < \infty$, such that $|x(-n)z_1^n| \leq M$ for all $n \geq 0$. But for any $z$ satisfying $|z| < |z_1|$, let $r = \frac{|z|}{|z_1|} < 1$, and

$$|A(z)| = |\sum_{n=0}^{\infty} x(-n)z^n|$$

$$\leq \sum_{n=0}^{\infty} |x(-n)||z^n| = \sum_{n=0}^{\infty} |x(-n)||z_1^n|r^n$$

$$\leq \sum_{n=0}^{\infty} Mr^n = \frac{M}{1 - r} < \infty$$

which must converge since $r < 1$.

Similarly,

$$B(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

converges for all $z$ outside some circle about the origin. (Replace $z$ with $1/z$ in the previous argument.) Hence the ROC has the form $R_1 < |z| < R_2$. If $R_2 < R_1$, then the $Z$-transform has empty ROC (and it doesn’t exist).
Fact: The ROC is bounded by the poles of $X(z)$. The inner circle is determined by the largest magnitude pole of the causal (right-sided) part of the signal; the outer circle is determined by the smallest pole of the anti-causal (left-sided) part of the signal.

Example 4. Find the $Z$-transform and ROC of the signal

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{4}{5}\right)^n u(n) - 3^n u(-1-n) - 4^n u(-1-n)$$
Z-Transform Properties

1. Linear

2. Delay: \( x(n - n_0) \xrightarrow{Z} z^{-n_0}X(z) \)

Application: linear difference equation systems – use the Z-transform to find the transfer function.

\[
y(n) = \sum_{k=1}^{N} a_k y(n - k) + \sum_{k=0}^{M} b_k x(n - k)
\]

\[
H(z) = \frac{Y(z)}{X(z)} = \]

Poles and zeros.
3. BIBO Stability: Assume that a discrete-time LTI system has impulse response \( h(n) \to H(z) \). The system is BIBO stable if and only if the unit circle lies within the region of convergence of \( H(z) \).

Proof.

ROC contains \(|z| = 1\) \(\Rightarrow\)

\[
|H(z)|_{|z|=1} = | \sum_{n=-\infty}^{\infty} h(n) z^{-n} | \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty
\]

Summary: BIBO stability of LTI system:

\[
\sum_{n=-\infty}^{\infty} |h(n)| < \infty \iff \text{ROC contains } |z| = 1
\]
Example 5. Find the Z-Transform of

\[ h(n) = \begin{cases} 
1, & 0 \leq n < N \\
0, & \text{otherwise}
\end{cases} \]

Find all finite poles and zeros. Let \( N = 4 \) and sketch the location of the poles and zeros in the \( z \)-plane. Determine the region of convergence. If \( h(n) \) is the impulse response of an LTI system, is the system BIBO stable?
Example 6. A difference equation system is described by
\[ y(n) + ay(n - 1) = x(n) - 0.5x(n - 1). \]

Find the transfer function, \( H(z) \).
Find all finite poles and zeros.
Sketch the ROC if the system is causal.
Sketch the ROC if the system is non-causal.
For what range of values, \( a \), is the causal system BIBO stable? For what range of values, \( a \), is the non-causal system BIBO stable?
Example 7. Find $X(z)$ for the signal

\[ x(n) = \begin{cases} 
na^n, & 0 \leq n \\
0, & n < 0
\end{cases} \]
Example 8. Find $X(z)$ for the signal

$$x(n) = \begin{cases} a^n, & 0 \leq n, \quad n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$
Example 9. A LTI system has transfer function

\[ H(z) = \frac{1 + z^{-1}}{1 - \frac{5}{2} z^{-1} + z^{-2}}. \]

a. Find all finite poles and zeros and sketch their location in the \( z \)-plane.

b. Shade the region of convergence for the three cases:
   
   i. \( H(z) \) is a causal system. Find the difference equation for the system.
   
   ii. \( H(z) \) is a non-causal system. Find the difference equation for the system.
   
   iii. \( H(z) \) is a mixed causal and non-causal system. Find the difference equation for the system.

c. For what type system is \( H(z) \) BIBO stable?
Example 10. A causal, LTI system has difference equation description

\[ y(n) = 1.4y(n - 1) - 0.74y(n - 2) + x(n) \]
\[ - 1.6x(n - 1) + x(n - 2) \]

a. Draw Direct Form I and II realizations.
b. Find the system transfer function, \( H(z) \).
c. Find all finite poles and zeros, sketch their location in the \( z \)-plane, and shade the region of convergence.