A discrete-time, linear time-invariant system with impulse response $h(n)$ is

a. Memoryless iff $h(n) = A \delta(n)$.

b. Causal iff $h(n) = 0$ for all $n < 0$.

Example: a discrete-time system has input-output description $y(n) = 2x(n) + 5$. Find the impulse response of the system. Is the system causal? Explain.


c. BIBO stable iff

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$
Two sets of discrete-time signal:

i) The set of absolute summable signals,

\[ l_1 = \left\{ x(n) : \sum_{n=-\infty}^{\infty} |x(n)| < \infty \right\}. \]

ii) The set of square summable signals,

\[ l_2 = \left\{ x(n) : \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \right\}. \]

Which set is larger? If \( x(n) \in l_2 \), is \( x(n) \in l_1 \)? Or vice-versa?

Consider, for example, \( x(n) = \begin{cases} \frac{1}{n}, & n \geq 1 \\ 0, & n \leq 0. \end{cases} \in l_1 \),

\[ \sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \quad \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{\pi^2}{6} < \infty. \]
Fact: For discrete-time linear, time-invariant systems, a necessary and sufficient condition for BIBO stability is

\[ \sum_{n=-\infty}^{\infty} |h(n)| < \infty. \]

Proof. First, what does the definition of BIBO stability mean? A system is BIBO stable if, for every bounded input, the output is bounded.

\[
\begin{array}{ccc}
  x(n) & \xrightarrow{T} & y(n) \\
  \downarrow & & \downarrow \\
  \text{T[x(n)]} & & \\
\end{array}
\]

i.e. for every time index \( n \), \( |x(n)| \leq M_x < \infty \) then for all \( n \), \( |y(n)| \leq M_y < \infty \), where \( M_x, M_y \) are finite bounds on the magnitudes of \( x(n), y(n) \), respectively.

i) Show \( \sum_{n=-\infty}^{\infty} |h(n)| < \infty \) \( \Rightarrow \) BIBO stable.

ii) Show the reverse:
\[ \sum_{n=-\infty}^{\infty} |h(n)| = \infty \Rightarrow \text{BIBO unstable}. \]
Since \( \sum_{k=0}^{\infty} |h(n-k)x(n-k)| \leq \sum_{k=0}^{\infty} |h(n)| |x(n-k)| \),

if \( x(n) \) is bounded, then \( |x(n)| \leq M < \infty \) for all \( n \)

\[
\left( \sum_{k=0}^{\infty} |h(n)| \right) M < \infty
\]

Show reverse: Find a bounded \( x(n) \) that causes \( y(n) \) to be unbounded.

Let \( x(n) = \begin{cases} \frac{h(n)}{|h(n)|} & \text{if } h(n) \neq 0 \\ \frac{1}{|h(n)|} & \text{if } h(n) = 0 \\ 0 & \text{if } h(n) = 0 \end{cases} \)

\[
y(n) = \sum_{k=0}^{\infty} h(n-k)x(n-k) = \sum_{k=0}^{\infty} \frac{h(n-k)}{|h(n-k)|} \frac{1}{|h(n-k)|}
\]

So \( y(n) = \infty \Rightarrow \text{unstable} \).
Examples:

a) Let $y(n) = 2x(n) + 1$. Find the impulse response. Is this system BIBO stable? $h(n) = 2\delta[n+1]

$$\sum_{n} h(n) = \infty$$

Note: Linear system $x(n-1) = \sum_{k=0}^{\infty} x(n-k) = \sum_{k=0}^{\infty} h(k)x(n-k)$

b) Let $y(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$, $0 < N < \infty$. Find the impulse response. Is this system BIBO stable?

$$h(n) = \begin{cases} \frac{1}{N}, & n=0,1,\ldots,N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{n} h(n) < \infty$$

So stable.

c) A LTI system has impulse response $h(n)$. Find the unit-step response in terms of the impulse response. Also, express the impulse response in terms of the unit-step response.

Let $s[n] = \sum_{k=0}^{\infty} h(k)x(n-k)$

$$= \begin{cases} 0, & n<0 \\ \sum_{k=0}^{n} h(k) \text{ if } n \geq 0 \end{cases}$$