Scaling: a canonical design problem for networks

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Scaling: a canonical design problem for networks

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Motivated by decentralized-control and numerical-methods applications, we explore the scaling problem. That is, we consider the design of a diagonal matrix $K$ so as to (i) place the eigenvalues of the product $KG$ in the OLHP and (ii) optimize a scalar performance measure $c(KG)$ of interest, for a square topology matrix $G$. We first illustrate our methodology for scaling design, which combines optimization machinery with graph-algebra notions, for a class of topology matrices and a specific performance measure. In particular, for symmetric positive definite topologies and for a dominant-eigenvalue-ratio performance measure, we find that the optimally-scaled system $KG$ has a special structure: the right eigenvectors associated with the largest and smallest eigenvalues have corresponding entries of identical magnitude and hence differ only in sign pattern. Using this structure, we develop both explicit and algorithmic methods for computing the optimal scaling and cost. After illustrating our methodology with this example, we characterize the optimally-scaled system for several relevant cost measures and classes of topology matrices. Finally, we give some first explorations on how these methods can be extended to address the more general eigenvalue-placement problem.

1. Problem formulation and motivation

The design of distributed algorithms that exploit a network or system’s topological (graph) structure is of growing interest in several controls application areas, including in autonomous vehicle coordination and sensor networking. Similarly, certain numerical-methods applications require exploiting the topological structure or sparsity pattern of a matrix to achieve fast computation. Several design tasks in these application areas can be distilled to the following linear-algebraic problem: given a matrix $G$ that represents a system’s topology, design a (block)-diagonal scaling matrix $K$ in order to (i) place all eigenvalues of the scaled system $KG$ in the ORHP and (ii) optimize a scalar performance measure $c(KG)$ that captures an important property of the scaled system or dynamics defined thereof. In this article, we formulate and motivate this canonical problem, and then develop a solution methodology. Our methodology has the further advantage of identifying structural features of the optimal scaled system, and permitting graph-theoretic characterization of the optimal performance and the optimal scaled system.

In the remainder of this section, we first give a general formulation of the design problem, and subsequently motivate the problem from several angles. In §2, we illustrate our solution methodology by optimizing a particular performance measure—the dominant eigenvalue ratio—over the class of positive diagonal scalings, for all symmetric $G$. In §3, we use this methodology to characterize optimal scalings with respect to several other cost metrics, and also to gain insight into the problem of eigenvalue placement through scaling.

1.1 General linear-algebraic problem formulation

Let us consider a set of $n$ matrices $G_1, \ldots, G_n$, where each $G_i$ has dimension $m_i \times n$. Because these matrices capture network topological and/or structural features in our applications of interest, we presciently refer to them as topology matrices. We also find it convenient to define a partitioned full topology matrix $G$, where $G = \begin{bmatrix} G_1 & \cdots & G_n \end{bmatrix}$. In a general sense, the full topology matrix represents internal structure of the system/network under study, and is assumed fixed.
The designer has the freedom to choose \( n \) gain vectors \( \mathbf{k}_1, \ldots, \mathbf{k}_n \), where \( \mathbf{k}_i \) is a vector of length \( m_i \). These gain vectors are to be chosen so that the scaled system \( A = KG, \mathbf{K} \equiv \text{diag}(\mathbf{k}_1^T) \), has all eigenvalues in the ORHP and further is optimal with respect to a scalar performance measure \( c(KG) \). We use the terminology gain matrix for the block-diagonal matrix \( K \).

The problem of placing the eigenvalues of a square matrix in the ORHP (or equiva-lently OLHP) through scaling by a diagonal matrix—which we refer to as stabilization by scaling—has been studied in a sequence of articles (Fisher and Fuller 1958, Ballantine 1970, Corfmat and Morse 1973, Roy et al. 2005), and is well-motivated in a range of applications. Fisher and Fuller (1958) contains a broad matrix-theoretic sufficient condition for the existence of a stabilizing scaling. Ballantine (1970) gives an alternate verification of this sufficient condition, which is based on root-locus methods. Motivated by discrete-time decentralized control applications, Corfmat and Morse (1973) give conditions under which the eigenvalues of \( I + KG \) can be placed within the unit disk using the diagonal scaling \( K \). Our recent work generalizes the original sufficient condition of Fisher and Fuller, using perturbation arguments (Roy et al. 2005).

Beyond achieving stabilization, it is natural to apply scaling to optimize a performance measure of interest; this performance optimization goal is the primary focus of this article. In many applications of interest to us, we are concerned with optimizing an associated linear dynamic system or linear iteration, within a set of constraints. Thus, we would expect to measure the performance of these systems based on the their state trajectories (e.g., the solutions of \( \dot{x} = -KGx \)), or in terms of parameters of the scaled system \( A = KG \) that give insight into these trajectories (dynamics). The following are specific performance measures of these types that we shall consider, presented along with classical motivations for their use in the numerical computations and control communities.

(i) The condition number, which we denote \( \delta_p(A) \), is the ratio of the largest and smallest singular values of a matrix \( A \). The condition number of a matrix \( A \) has classically been used as a measure for the accuracy of the iterative solution of the set of algebraic equations \( Ax = b \), as well as for the speed of convergence of this iteration (see e.g., Greenbaum (1997)).

(ii) It is classical that the locations of eigenvalues of a matrix \( A \) play a critical role in the dynamics of \( \dot{x} = -Ax \) and the solution of the associated algebraic equations. Of particular importance, for matrices with real and positive eigenvalues, the ratio of the largest to the smallest eigenvalue can be used to characterize convergence properties of associated dynamics/iterations and (from a controls viewpoint) gives insight into the eigenvalue placement problem. We refer to this ratio as the dominant eigenvalue ratio of a matrix \( A \), and use the notation \( \lambda_d(A) \) for it.

(iii) Other eigenvalues (or ratios of these eigenvalues to the one of smallest magnitude) are also considered as performance measures, because of the importance of eigenvalue placement in control-system performance.

(iv) For stable linear systems with real eigenvalues, the convergence rate is dominated by the eigenvalue of the system matrix \( A \) that is closest to 0 in magnitude. Thus, maximization of this minimum eigenvalue \( \lambda_1(A) \) is a worthwhile design goal. Of course, for our problem, \( \lambda_1(A) \) can be made arbitrarily large by multiplying the gain matrix \( K \) by a constant. In reality, however, the sizes of the gains are limited by power, actuation, or other constraints, and hence it is more sensible to normalize the minimum eigenvalue in measuring performance. In particular, motivated by possibility for actuator saturation, we consider the performance measure \( c(A) = \lambda_1(A)/\|A\|_\infty \).

(v) Lyapunov techniques are also classically used for performance and robustness analysis of controller designs. In particular, for a particular matrix \( A \), let us say that we can find a matrix \( P > 0 \) (i.e., a positive definite matrix \( P \)) such that \( A^TP + PA > 0 \). Then the system \( \dot{x} = -Ax \) is known to be stable, and further several performance characteristics of the system (for example, robustness) are known to be governed by the Lyapunov exponent \( \gamma_p(A) = \sqrt{\lambda_1((A^TP + PA))/\lambda_d(P)} \). We shall thus be interested in properly choosing a Lyapunov matrix \( P \) and designing the gain matrix \( K \) to maximize \( \gamma_p(KG) \). Of course, in cases where \( P \) and \( K \) can be chosen to demonstrate stability, the Lyapunov exponent can made arbitrarily large through scaling of \( K \), and hence we again require normalized measures. Here, the measure that we shall consider is \( s(A) \equiv \gamma_p(A)/\|A\|_\infty \). We consider optimization of \( s(A) \) over \( K \), and over the class of diagonal Lyapunov functions.

(vi) There are a range of other pertinent scalar performance measures, perhaps most notably quadratic cost measures. We do not pursue these other measures explicitly in this article, but plan to apply the methodologies developed here to these measures in the future.

Our aim is to design \( K \) to place the eigenvalues of \( KG \) in the ORHP and to optimize one of these
performance measures. More broadly, we shall address the following three design questions.

- How should $K$ be designed to optimize one or more of these performance measures?
- What are structural properties of the the scaled system, for optimal or near-optimal choices of $K$?
- What characteristics are required of the full topology matrix $G$, such that desirable values of the performance measure(s) can be obtained through appropriate scaling?

In addressing these design problems, we shall sometimes find that our results are only applicable to certain classes of full topology matrices. Thus, let us delineate several classes of full topology matrices.

(i) Positive definite symmetric topologies are ones in which the full topology matrix is a (square) symmetric positive definite matrix, with each topology matrix having dimension $1 \times n$. Positive definite symmetric topologies include grounded Laplacian topologies, which can represent relative-position or other relative state measurements, and hence can be used to model observations/communication in autonomous vehicle teams.

(ii) Arbitrary single-observation topologies are ones in which each topology matrix has dimension $1 \times n$, and hence the full topology matrix $G$ is square (we use the term observation with the vehicle control and sensor network applications in mind). Such models are appropriate for various controller/algorithm-development tasks, in which each agent in the network makes a scalar observation.

(iii) Multi-observation topologies are ones in which each topology matrix may be of arbitrary dimension. Our earlier work has clarified that agents in sensor networks and autonomous vehicle teams can exploit multiple observations to achieve stabilization under broad conditions, and hence we might expect such multi-observation topologies to permit higher performance also.

Holistically, we notice that the design problem is parametrized solely by the (appropriately-partitioned) full topology matrix $G$, as well as as a particular performance measure $c()$, and hence we describe a particular design with these two parameters.

1.2 Vehicle control, sensor networking, and other control applications

The emergence of autonomous vehicle control and distributed sensor networking applications has fostered new interest in the development of decentralized controllers and algorithms (see e.g., Stilwell and Bishop (2000), Fax and Murray (2003), Roy et al. (2004) and Xiao and Boyd 2004)). The algorithms/controllers for these sensing network applications have in common that they must coordinate the efforts of agents with simple internal dynamics, local observation capabilities, and severely constrained actuation, to achieve a complex and global task. Several promising studies have identified that the observation topology (graph) of the network plays a critical role in algorithm/controller development, and have developed conditions for the existence of controller/algorithms that can complete a range of tasks (e.g., Fax and Murray (2003) and Roy et al. (2004, in press)). However, we contend that the constructive design of good or optimal controller/algorithms remains a daunting challenge (see Dunbar and Murray (2002) and Xiao and Boyd (2004) for relevant discussions), and that the significance of the observation graph with respect to the design process and the optimal closed-loop performance is not well-understood. Of particular interest to us, Xiao and Boyd (2004) develop an LMI-based methodology for designing the edge weights of a Laplacian matrix to maximize the subdominant eigenvalue, but their approach does not give explicit insight into the structure of the optimized system.

In a general sense, the recent research on vehicle and sensor coordination builds on classical work on decentralized control, and hence it is worth our while to connect our results to this broader literature. The seminal work of Davison and Wang (1973) uses perturbation arguments to give necessary and sufficient condition for the existence of a stabilizing dynamic LTI controller for a general decentralized linear system in terms of the notion of a fixed mode. Corfmat and Morse (1976) provide a structural interpretation with a graph-theoretic flavour for this condition. Of particular interest to us, broad sufficient conditions for decentralized static stabilization have also been developed (see Fisher and Fuller (1958), Ballantine (1970), Corfmat and Morse (1973) and Roy et al. 2005). However, while the existence of decentralized controllers has been well-characterized through these works, their perturbation-based arguments do not directly suggest good controller designs. We view our work as a first result in high-performance design for decentralized systems, albeit in the context of a very simple system model (which is applicable to various autonomous vehicle control, sensor networking, and numerical computation problems). Also of interest in the classical decentralized control literature, Khalil and Geromel (1982) give algorithms for checking the stability of a system over the class of diagonal Lyapunov functions. Our current work builds on theirs, in that we identify structural features of $A^TP+PA$ in the case where a diagonal $P$ is chosen to optimize the Lyapunov exponent.
From the broader perspective of decentralized control, the recent work on vehicle control and sensor networking has exploited the especially structured representations for these systems (representations with limited internal dynamics but significant topological interaction) to develop explicit graph-theoretic conditions for the existence of stabilizing controllers (e.g., Fax and Murray (2003) and Roy et al. (2004)), and for performance evaluation of particular controllers. Since we also consider these highly structured systems, we are able to give structural and graph-theoretic insights into the dynamics of the optimally scaled system, and into the optimal scaling itself. In particular, we find that optimal design with respect to several relevant performance measures enforces special structure on the eigenvectors of the scaled system. This special structure in turn allows the development of finite-dimensional algorithms for design, as well as graph-algebra characterizations of the optimum.

Let us now show that a basic problem of interest in vehicle coordination can be distilled to the design problem developed above. In particular, let us consider the problem of formation using static control, for networks of agents with single- and double-integrator internal dynamics and decentralized sensing capabilities. Several researchers in recent years have developed existence conditions for formation in networks of decentralized sensing agents (e.g., Fax and Murray (2003) and Roy et al. (2004)). That is, they have given conditions on the sensing network topology such that a network of agents can be controlled to form a (fixed or moving) pattern in space. The problem of developing high-performance formation controllers remains an open one, and hence design with respect to a performance measure is of interest in this context.

In Roy et al. (2004), we have shown that a control strategy that achieves stabilization also achieves formation (through use of an appropriate reference signal). We have discussed how agents moving in multiple spatial directions can be represented using multiple scalar agents. We have also shown that, when static control is used, a high-gain control strategy can achieve good performance for a double-integrator network if a good controller can be designed for the single-integrator network and position and velocity observations are both available, i.e., formation of agent positions can be achieved whenever formation of agent velocities is achieved (see Roy et al. (2004) for details). Hence, without loss of generality, the formation problem for single-integrator networks can be viewed as the following stabilization problem: consider a network of $n$ agents, $1, \ldots, n$. Each agent $i$ has internal dynamics $x_i = u_i$, where $x_i$ represents the (scalar) state of agent $i$ and $u_i$ is the scalar control input for the agent. For convenience, we assemble the agents’ states into a state vector $x$, where $x^T \triangleq [x_1 \ldots x_n]$ and assemble the inputs into an input vector $u$, where $u^T \triangleq [u_1 \ldots u_n]$. The agent $i$ in general makes $m_i$ observations, each of which is assumed to be a linear combination of the positions of other agents at the current time. That is, agent $i$ makes observations $y_i = G_i x$, where $G_i$ is a matrix of dimension $m_i \times n$. Each agent determines its control input from its observations using a static linear time-invariant controller: $u_i = k^T_i y_i$, where $k_i$ is a length-$m_i$ vector. Assembling the dynamics of the $n$ agents and substituting for the feedback control input, we immediately find that the closed-loop dynamics are given by $\dot{x} = K G x$, where $G^T \triangleq [G_1^T \ldots G_n^T]$ and $K \triangleq \text{diag}(k_i^T)$. Our aim is to choose the block-diagonal matrix $K$, in such a manner that the closed-loop system $\dot{x} = K G x$ is a stable one, and in fact approaches the origin in a desirable way (e.g., sufficiently quickly, along a trajectory that minimizes a quadratic cost, or along a trajectory that remains within a desirable subset of $R^n$). Thus, we see that the controller design problem can be viewed as an example of the problem formulated above, with the full topology matrix $G$ capturing the network’s observation topology, and with one of the performance measures described before (e.g., the quadratic cost measure, a Lyapunov exponent, or the dominant eigenvalue ratio).

In discussing the application of our work to high-performance decentralized controller design, we must stress that our results here only constitute a first step towards design. In particular, much further work is needed to (i) adapt these methods to general linear system and controller models, (ii) characterize the full scope of dynamics permitted by decentralized control (rather than optimizing one measure), and (iii) incorporate such important model features as actuator saturation. Nevertheless, we believe that the methodology presented here will be a valuable tool in addressing these more general problems, in addition to directly allowing controller development for the simple single- and double-integrator networks introduced above.

### 1.3 Preconditioning and other computational applications

In a parallel direction, iterative structurally-localized or sparse computation can be used in some applications as a tool for analyzing large-scale systems with low complexity. For instance, diagonal and block-diagonal preconditioners—which can be viewed as achieving partial or local combination/scaling of system equations—have long been used in solving linear systems (see e.g., Greenbaum (1997)). Distributed algorithms have also been used in such computational tasks as graph partitioning (Wan et al. 2005) and numerical solution of finite-element problems. In these applications, it is typical that a particular algorithm is proposed (e.g., an often-used diagonal preconditioner is one in which the
entries are the inverses of the system’s diagonal entries) and characterized, in some cases in terms of system graph properties. Our viewpoint is that it is worth characterizing these numerical problems over a class of distributed/localized algorithms, and in turn finding optimal algorithms and their structures.

For illustration, let us formulate a classical preconditioner-design problem in our framework. Specifically, let us consider solving the set of linear equations $Gx = b$, where $G$ is symmetric and positive definite, using a conjugate-gradient method. If the matrix $G$ is ill-conditioned, i.e., if it has large condition number, the solution through conjugate gradient methods is slow and subject to error. In such cases, it is classical to instead solve the equations $(Q^T G Q)y = Q^T b$ for $y$ and then find $x = Qy$, where the preconditioner $Q$ is chosen so that the $Q^T G Q$ is better-conditioned than $G$. Often, positive diagonal and block-diagonal preconditioners $Q$ are used, because these can be easily and accurately found and also require little effort to multiply with other vectors/matrices. Traditionally, a particular preconditioner $Q$ is chosen and its performance analysed, but let us instead address the problem of optimally designing a diagonal preconditioner $Q$. Specifically, assuming a diagonal preconditioner $Q \triangleq K^{1/2}$, we find that the preconditioner-design problem is equivalent to the problem of selecting $K$ to minimize the condition number $\tilde{\sigma}_n(K^{1/2} G K^{1/2})$. Noticing that $(K^{1/2} G K^{1/2})$ is symmetric and positive definite and using a similarity argument, we find that equivalently $K$ should be designed to minimize the dominant eigenvalue ratio $\tilde{\sigma}_n(KG)$. Hence, we have shown the preconditioner design problem is in fact an example of the design problem formulated above, with the our methodology thus has the potential for finding good diagonal preconditioners, and identifying the structure of the preconditioned system when such a preconditioner is used. We note that Boyd et al. (1994) and coworkers have developed numerical (in particular, LMI-based) optimizations of such condition numbers over the class of diagonal scalings, but structural characteristics of the optimal scaling and scaled system have not been determined.

### 1.4 Connections to algebraic graph theory

Because we are able to develop graph-theoretic characterizations of the optimum, our results hold potential as a contribution to the field of algebraic graph theory (see Chung (1997) for a thorough introduction). Broadly, this field is concerned with relating algebraic properties of a matrix (e.g., the second-smallest eigenvalue of a Laplacian matrix) with properties of a graph associated with the matrix (e.g., the minimum cut of such a graph). These relationships prove valuable in several application areas, including for generating bounds for combinatorial graph theory problems and for characterizing the settling properties of certain linear dynamics defined on a graph. Of particular interest to us, certain scalings of Laplacian matrices have been considered, with the motivation that algebraic properties of the scaled system may give new insight into the graph (Chung 1997). (For instance, a classical result is that the determinant of a combinatorial Laplacian equals the number of spanning trees in the associated graph.)

To the best of our knowledge, there is only a little work that attempts to relate properties of a graph to algebraic properties of a class of matrices (rather than a single matrix) defined from that graph. In this domain, Chung gives bounds on the chromatic number in terms of the maximum eigenvalue over the class of all possible edge-weighted Laplacians with a given node-edge topology, and relates this eigenvalue maximum to the Lovasz function from information theory (Chung 1997). In a similar vein, we consider the class of all scaled versions of an arbitrary (weighted) symmetric graph, and connect algebraic properties of an optimum within that class to a hard-to-find graph-theoretic quantity (in our case, the maximum cut in the graph). Our analysis is also potentially of value from an algebraic-graph theory perspective because it gives insight into the eigenvector structure of the optimally scaled system, and can provide stronger bounds on hard-to-find graph properties. We note that the the optimal scaling (and hence bounds on graph-theoretic properties) can be explicitly found with relatively low cost through our methods.

### 1.5 Overview of the design methodology

Our aim in this paper is to illustrate a methodology for solving the design problem posed above, and hence develop high-performance distributed algorithms/controllers for interesting canonical network tasks. Specifically, using optimization arguments together with algebraic characterizations of the performance measures, we identify structural properties of the optimally-scaled system (e.g., relationships between eigenvector entries) and in turn obtain the optimal scaling itself, for several performance measures and classes of graph topologies. This methodology further permits us to bound the optimal performance with respect to features of the network’s graph.

In the interest of space, here we use one performance measure—the dominant eigenvalue ratio—and one class of topology matrices—the symmetric class—to illustrate our methodology. Our motivation for using this measure is two-fold: first, the dominant eigenvalue ratio has strong connections to both numerical analysis and controller design problems, and can be related to the other performance measures, so its analysis is
perhaps most directly applicable to the problems discussed here. Second, we have a complete analysis in this case, so that its presentation fully illustrates the methodology.

2. Illustration of the design method: the dominant eigenvalue ratio

We address the design problem with a symmetric positive definite matrix full topology matrix and the dominant eigenvalue ratio performance measure \( \hat{\lambda}_n(KG) \). We begin by finding the set of \( K \) such that the eigenvalues of \( KG \) are in the ORHP, and also giving some characterizations of the eigenvalues of \( KG \) and the dominant eigenvalue ratio over this set of gain matrices. We then exploit the notion of eigenvalue sensitivity to gain insight into the scaled system that achieves the minimum dominant eigenvalue ratio. We then give graph-theoretic bounds on the dominant eigenvalue ratio, and conclude by developing bounds on the extremal values of other performance measures in terms of the minimum dominant eigenvalue ratio.

2.1 Preliminary analysis of the scaled system's eigenvalues

We begin with some general results of the eigenvalues of \( KG \), where \( K \) is diagonal and \( G \) is symmetric. These results readily follow from classical results on the products of symmetric positive definite matrices and standard similarity transform arguments, and so we omit the proofs.

First, let us delineate gain matrices \( K \) such that \( KG \) has eigenvalues in the ORHP, and give some further characterization of the eigenvalues in this case.

**Theorem 1:** Consider the product \( KG \), where \( G \) is symmetric positive definite and \( K \) is diagonal. The eigenvalues of \( KG \) are all in the ORHP if and only if \( K > 0 \) (i.e., if and only if the diagonal entries of \( K \) are strictly positive). Furthermore, if \( K > 0 \), the eigenvalues of \( KG \) not only are in the ORHP but are real and simple.

Since we wish to consider designs in which all eigenvalues have the same sign, we shall henceforth constrain \( K \) to be positive, i.e., we will optimize the performance over the class of positive scalings. It is also useful for us to note that the dominant eigenvalue ratio is invariant to scaling by a constant: \( \hat{\lambda}_n(KG) = \hat{\lambda}_n(cKG) \), for any \( c > 0 \).

Let us conclude our preliminary analysis by giving some characterization of the interval on the real axis where the dominant eigenvalue ratio can be placed using scaling. Since the eigenvalues of \( KG \) are constrained to be positive and real in our formulation, we notice that the dominant eigenvalue ratio \( \hat{\lambda}_n(KG) \) is necessarily a real number on the interval \([1, \infty)\). In fact, we claim that \( \hat{\lambda}_n(KG) \) can only be placed in an interval \([\alpha, \infty)\), where \( \alpha > 1 \), unless \( G \) is diagonal. That is, we can make the dominant eigenvalue ratio as large as desired, but can only reduce it to a non-trivial minimum greater than 1. The following theorem provides a careful statement of this claim.

**Theorem 2:** Consider scaling a symmetric positive definite matrix \( G \) by a positive definite diagonal matrix \( K \). Then the dominant eigenvalue ratio of the product, \( \hat{\lambda}_n(KG) \), is a real number that can be placed anywhere in a range \([\alpha, \infty)\) by designing \( K \). The lower bound \( \alpha \) for the dominant eigenvalue ratio is equal to 1 if and only if \( G \) is diagonal, and is strictly greater than 1 otherwise.

The following re-statement of the lower bound in Theorem 2 is valuable, because it gives an eigenvalue-placement interpretation to the result.

**Corollary 1:** Consider a positive definite matrix \( G \). Then a gain matrix \( K \) can be chosen to put all eigenvalues of \( KG \) at one location, if and only if \( G \) is diagonal.

Theorem 2 identifies a significant constraint in eigenvalue placement when diagonal scaling is used. Unlike in the case where \( K \) can be dense, scaling by a diagonal matrix only permits the dominant eigenvalue ratio to be reduced to a minimum that is in general greater than 1. For our applications of interest, the dominant eigenvalue ratio typically must be made small. Hence, characterizing the minimum dominant eigenvalue ratio is a worthwhile design objective. The ensuing analysis will identify this extremal value for the performance measure, and the scaling that achieves it.

2.2 Structure of the optimal scaled system

Our goal is to design \( K \) to minimize the dominant eigenvalue ratio, subject to the constraint that \( KG \) has eigenvalues in the ORHP. Thus, for the remainder of this section, we assume \( K > 0 \).

In describing this analysis, we find it convenient to introduce some further notation and terminology:

(i) We define \( \alpha \) to be the minimum achievable dominant eigenvalue ratio over the class of diagonal scalings \( K > 0 \). That is, \( \alpha \triangleq \inf_{K > 0} \hat{\lambda}_n(KG) \). We often refer to \( \alpha \) as the minimum dominant eigenvalue ratio.

(ii) We use the notation \( K^* \arg\inf_{K > 0} \hat{\lambda}_n(KG) \), and refer to \( K^* \) as a minimizing gain matrix. (To be more precise, we use this notation for any gain matrix that minimizes the dominant eigenvalue ratio).
(iii) We refer to a scaled system $K^* G$, i.e. a scaled system for which the gain matrix is a minimizing one, as an extremal scaled system. 
(iv) We define the set $\mathcal{S}$ to contain all diagonal $n \times n$ matrices whose diagonal entries are $\pm 1$. We note that $\mathcal{S}$ has a cardinality of $2^n$.
(v) We refer to the smallest eigenvalue of a matrix $A$ (with positive real eigenvalues) as $\lambda_1(A)$ and the largest eigenvalue as $\lambda_n(A)$.

We begin with a theorem that characterizes the eigenvectors of an extremal scaled system. This structural insight into the extremal system is derived using an optimization argument together with eigenvalue-sensitivity notions.

**Theorem 3:** Assume that $G$ is symmetric, and consider any extremal scaled system $K^* G$. The extremal scaled system has a right eigenvector $v_1$ associated with $\lambda_1(K^* G)$ and a right eigenvector $v_n$ associated with $\lambda_n(K^* G)$ that are related by $v_n = S v_1$, where $S \in \mathcal{S}$.

**Proof:** To streamline the proof, let us assume that $\lambda_1(K^* G)$ and $\lambda_n(K^* G)$ are non-repeated eigenvalues. The general case can straightforwardly be proved from this case using continuity arguments.

It is easy to check that any optimal gain matrix is strictly within the first quadrant (i.e., $K^* > 0$), and hence that the infimum in the expression for $\alpha$ is in fact a minimum. Thus, we know that the sensitivity of $\lambda_n(G)$ to any change in $K$ is zero, for $K = K^*$. That is,

$$ \frac{d}{dk_i} \lambda_n(G) |_{k^*} = 0, \quad (1) $$

for $i = 1, \ldots, n$. Noting that $\dot{\lambda}(G) = \dot{\lambda}_n(G) / \lambda_1(G)$, we can rewrite equation (1) as

$$ (\dot{\lambda}_1(G) \frac{d\lambda_n(G)}{dk_i} - \lambda_n(G) \frac{d\dot{\lambda}_1(G)}{dk_i}) |_{k^*} = 0. \quad (2) $$

Noticing that the eigenvalues of $G$ are simple and applying the eigenvalue sensitivity formulae, we obtain that

$$ w_{ni}(g_i^T v_n) \dot{\lambda}_1(K^* G) - w_{ni}(g_i^T v_1) \lambda_n(K^* G) = 0, $$

for $1 \leq i \leq n$. With just a little bit of algebra, we can reduce these equalities to $w_{ni} v_{ni} - w_{1i} v_{1i} = 0$, $1 \leq i \leq n$.

We then exploit the similarity of $G$ and $K^{1/2} G K^{1/2}$ to relate $v_n$ with $w_{ni}$ and $v_1$ with $w_{1i}$. Doing so, we find that $w_{ni} = (1/k_i) v_n$ and $w_{1i} = (1/k_i) v_1$. Substituting, we finally find that $v^*_n = v^*_1$, or equivalently that $v_n = \pm v_1$, for each $i$. Hence, the theorem is proved. Thus, we have shown that the eigenvectors associated with the largest and smallest eigenvalues of any extremal scaled system have entries of identical magnitude and different sign. We note that the same relationship holds between the eigenvectors of $K^{1/2} G K^{1/2}$ associated with the largest and smallest eigenvalues.

### 2.3 Algorithms for designing the optimal gain matrix

The next theorem gives a finite-dimensional algorithm for finding the minimum dominant eigenvalue ratio as well as identifying a minimizing gain matrix.

**Theorem 4:** The minimum dominant eigenvalue ratio is given by

$$ \alpha = \max_{S \in \mathcal{S}} \frac{\lambda_n(G^{-1/2} S G S^{-1/2})}{\lambda_1(G^{-1/2} S G S^{-1/2})}, \quad (3) $$

where $G^{1/2}$ is the positive definite symmetric square root of $G$. Furthermore, a minimizing gain matrix has diagonal entries given by $k_i^* = (G^{1/2} b_i^*) / (G^{1/2} b^*)$, where $b^*$ is the right eigenvector of $\lambda_n(G^{-1/2} S G S^{-1/2})$ and where $S^* = \arg \max_{S \in \mathcal{S}} \lambda_n(G^{-1/2} S G S^{-1/2})$.

**Proof:** From similarity, we see that $\alpha = \min_{K > 0} (\lambda_n(K^{-1/2} G K^{1/2}) / \lambda_1(K^{-1/2} G K^{1/2}))$. Since $K^{1/2}$ is symmetric and positive definite, we can develop expressions for its maximum and minimum eigenvalues using the Courant–Fisher theorem (e.g., Chung (1997)). Thus, we can rewrite $\alpha$ as

$$ \alpha = \min_{k > 0} \frac{\max_{q \in \mathcal{Q}} \langle q^T G q \rangle / \langle q^T K^{-1} q \rangle}{\min_{r \in \mathcal{R}} \langle r^T G r \rangle / \langle r^T K^{-1} r \rangle} \quad (4) $$

We can further rewrite equation (4) as

$$ \alpha = \min_{k > 0} \frac{\max_{q \in \mathcal{Q}} \langle q^T G q \rangle / \langle q^T K^{-1} q \rangle}{\min_{r \in \mathcal{R}} \langle r^T G r \rangle / \langle r^T K^{-1} r \rangle} \quad (5) $$

However, for a minimizing $K$, the maximizing $q$ and the minimizing $r$ in the expression above are related by $q = Sr$, where $S \in \mathcal{S}$. Hence, we can limit our search to ratios of the form $(q^T G q / q^T K^{-1} q) / (r^T G r / r^T K^{-1} r)$ where $K$ is chosen so that $q = Sr$. That is, we can find $\alpha$ as

$$ \alpha = \min_{k \in K \in \mathcal{S}} \frac{r^T G r / r^T K^{-1} r}{r^T G r / r^T K^{-1} r}, \quad (6) $$

where $K$ comprises the set of all positive diagonal $K$ such the the eigenvectors $v_n$ and $v_1$ are related by $v_n = S v_1$ for some $S \in \mathcal{S}$. However, we notice that $SK = S K^{1/2} K^{-1/2} = S K^{-1}$, so $r^T S K^{1/2} S r = r^T K^{-1} r$. The expression for $\alpha$ thus simplifies to

$$ \alpha = \max_{S \in \mathcal{S}} \frac{r^T S G r}{r^T G r}, \quad (7) $$

where we are able to eliminate the minimization over $K$ altogether because the quantity to be minimized does not depend on $K$. Finally, noting that $G$ has a positive definite symmetric square root, we can
Let $K_i$, $i = 1, 2, 3, \ldots$ be a sequence of gain matrices, and in turn find the eigenvectors $v_1$ and $v_n$. In particular, notice that the vector $r$ that achieves the maximum in equation (7) is $r^* = G^{-1/2}b^*$. In turn, we see that the vector $z$ that minimizes the denominator of equation (4) is $z^* = (K^*)^{-1/2}r^* = (K^*)^{-1/2}G^{-1/2}b^*$. Substituting in the eigenvalue/eigenvector equation for $(K^*)^{-1/2}G(\xi)^{1/2}$ and doing some algebra, we recover that $k^*_i = (\xi_1)(G^{-1/2}b^*)/(G^{-1/2}b^*)$, for each $i \in [1, n]$. We do not know $\xi_1$, but of course scaling $K$ does not change the dominant eigenvalue ratio, so we can choose $k^*_i = (G^{-1/2}b^*)/(G^{-1/2}b^*)$ to achieve the minimum dominant eigenvalue ratio. Once we have determined the minimizing gain matrix, we can determine the eigenvalues and eigenvectors of $K^* G$ directly through eigenanalysis.

We have thus developed an algorithm, using which the minimum achievable dominant eigenvalue ratio and the scaling that achieves it can both be found. In fact, the algorithm we have developed gives an exact answer in finite time: specifically, the product $G^{-1/2}SGSG^{-1/2}$ must be found and its dominant eigenvalue/eigenvector evaluated for the $2^n$ matrices in $\mathcal{S}$. For certain applications, such as some autonomous-vehicle control ones, the number of agents $n$ is small and hence we can straightforwardly apply this algorithm to controller to minimize the dominant eigenvalue ratio. Furthermore, for vehicle control and sensor network applications (among others), the dominant cost is usually related to the physical dynamics of the system rather than the computation required for design, and so some complexity in designing high-performance algorithms is well worth the cost. However, for such applications as preconditioning, the algorithm in its current form is not practical: evaluation of equation (3) requires inversion of $G$ for each $S \in \mathcal{S}$, so clearly the algorithm is too expensive for preconditioner design. With this computational shortcoming in mind, we also develop an iterative algorithm for finding the optimal gain matrix.

**Theorem 5:** Let $K_i$, $i = 1, 2, 3, \ldots$ be a sequence of gain matrices, and choose $K_1 = I$. Consider the following gradient-descent algorithm for constructing the sequence of gain matrices: $K_{i+1} = K_i - \nabla_{K_i} \lambda_n(K_i G)_K \Delta$, where $\Delta$ is small positive constant. For sufficiently small $\Delta$, we find that $K_i > 0$ for each $i$ and further that $\lim_{i \to \infty} \lambda_n(K_i G) \lambda_n(K_i G)$ exists and equals $\alpha$. That is, the algorithm finds a positive gain matrix that achieves the minimum dominant eigenvalue ratio.

**Proof:** We only sketch the proof of this theorem, in the interest of space. From Lipschitz-continuity arguments, one can show that $\lambda_n(K_i G)$ is a decreasing function of $i$, for sufficiently small $\Delta$. Since $\lambda_n(K_i G)$ is decreasing for sufficiently small $\Delta$, we can also show that $K_i$ cannot change sign and hence $K_i > 0$ for all $i$.

To continue, we notice that $\lambda_n(K_i G)$ is a strictly decreasing sequence with a finite lower bound $\alpha$, and hence $\lim_{i \to \infty} \lambda_n(K_i G) = \alpha$. Since the limit, $\lim_{i \to \infty} \nabla_{K_i} (K_i G)_K = 0$ exists, which in turn implies that $v_n \to S v_1$ for some $S \in \mathcal{S}$. However, from the proof of Theorem 4, it is clear that in fact $S \to S^*$ and $\lambda_n(K_i G)$ approaches $\alpha$ as $v_n \to S v_1$.

Our explorations suggest that the computational complexity of this algorithm is quite moderate. We note that the dominant computational cost at each stage of the algorithm is calculation of the eigenvectors of $K G$ associated with the smallest and largest eigenvalues. For many classes of graph matrices $G$, the eigenvectors of $KG$ associated with the largest and smallest eigenvalues can be found in $O(n^2)$. Furthermore, our explorations suggest that only a small number of iterations are needed to obtain a good estimate of $K^*$. We leave it to future work to carefully bound this convergence rate, and to fully explore the applicability of the algorithm for preconditioning. It is also worth noting that LMI-based methods can also be used to numerically obtain the optimum (Boyd et al. 2004).

### 2.4 A graph-theoretic bound on the dominant eigenvalue ratio

We also wish to identify how the performance of the optimized system depends on the topological structure. Based on this motivation, we develop a graph-theoretic characterization of the optimal gain matrix and corresponding scaled system.

In describing the result, we find it convenient to define/review some further terminology. Let us again consider an $n \times n$ symmetric positive definite full topology matrix $G$. We define the conductance graph $\Gamma(G)$ to be an undirected, weighted graph with $n + 1$ nodes, labeled $0, \ldots, n$. $\Gamma(G)$ has an edge between node $i$ and node $j$, if $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$, if $g_{ij} \neq 0$, and the weight of the edge is given by $-g_{ij}$. Also, there is an edge between node $i$ and node $0$ (which we call the ground node) if $\sum_{j=1}^{n} g_{ij} \neq 0$, and the weight of this edge is given by $\sum_{j=1}^{n} g_{ij}$. We notice that edge weights in this graph might be negative. Next, consider a partition of the nodes of $\Gamma(G)$ into two sets $A$ and $B$. We define $\text{Cut}(G, A, B)$ as the sum of the weights on the edges with
one end in A and one end in B. Also, consider the subgraph of G induced on a set of nodes A. We shall refer to the maximum over all partitions (y, v) of A of \(\text{Cut}(A, y, v)\) as the maximum two-cut of A, denoted \(\text{MaxCut}(A)\).

In this notation, we can obtain the following relationship between the dominant eigenvalue ratio and graph-theoretic constructs:

**Theorem 6:** Consider a symmetric positive definite matrix G, and let A and B be any partition of the nodes of \(G\) with B containing the ground node. Then the maximum dominant eigenvalue ratio \(\alpha\) satisfies

\[
\alpha \geq 4\text{MaxCut}(A)/\text{Cut}(G, A, B) + 1.
\]

We conclude our illustration by noting that the above results in turn permit us to develop bounds for one performance measure with respect to another. For instance, we can straightforwardly bound the extremal values of the quadratic cost measure in terms of the minimum dominant eigenvalue ratio. Since our aim here is to illustrate the optimization machinery, we do not pursue these bounds further here.

### 3. Other performance measures

In this section, we present structural insights into, and algorithms for finding, optimal scalings with respect to several other performance measures. This collection of results is largely presented without proof, in the interest of space and because of the similarity in the proofs with those for the dominant eigenvalue ratio measure. We present or discuss optimal scalings for the following performance measures and associated classes of topology matrices:

- the condition number, for arbitrary full-rank single-observation topology matrices;
- the cost \(\lambda_i((KG)K)^T/\|KG\|_\infty\) for symmetric positive definite topology matrices;
- a Lyapunov exponent-based cost, for arbitrary topology matrices.

Additionally, we present some insights into scaling for the purpose of eigenvalue placement, by considering optimization of multiple eigenvalue ratios.

#### 3.1 The condition number

Our aim is to determine the diagonal scaling \(K\) that minimizes the condition number \(\sigma_i((KG)K)/\sigma_1((KG)K)\). We recall that \(i\)th singular value of \(KG\) is equal to the square root \(i\)th eigenvalue of the matrix \((KG)K)^T\). Hence, we can phrase the design task as the problem of finding a diagonal scaling \(K\) such that \(\lambda_\alpha((KG)K)^T)/\lambda_1((KG)K)^T = \lambda_\alpha(KGG^TK)/\lambda_1(KGG^TK)\) is minimized. From similarity, we can further restate the design problem as that of finding \(K\) such that \(\lambda_\alpha(KGG^TK)/\lambda_1(KGG^TK)\) is minimized. Noting that \(K^2\) can be set to any positive diagonal matrix and that \(GG^T\) is symmetric and positive definite, we see that this design problem is in fact equivalent to the dominant-eigenvalue-ratio design problem. We can thus directly obtain structural results on the optimal solution, as well as finite-dimensional algorithms for finding the solution. Here are the significant results:

**Theorem 7:** Let \(u_\alpha(KG)\) be the singular vector of \(KG\) associated with the largest singular value, and let \(u_1(KG)\) be the singular vector associated with the smallest singular value (for convenience of presentation, let us assume that these singular values are unique; the theorem can be simply generalized in the same way as for the dominant eigenvalue ratio). For the scaling \(K = K^*\) that minimizes the condition number, \(u_\alpha(KG) = Su_1(KG)\), where \(S \in S\). That is, each entry of \(u_\alpha(KG)\) has the same magnitude but possibly different sign as the corresponding entry of \(u_1(KG)\).

**Theorem 8:** The minimum condition number is given by

\[
\mu = \sqrt{\max_{S \in S} \lambda_\alpha((GG^T)^{-1/2}SGG^TS(GG^T)^{-1/2})},
\]

where \((GG^T)^{1/2}\) is the positive definite symmetric square root of \(GG^T\). Furthermore, a minimizing gain matrix has diagonal entries given by \(k^*_l = ((GG^T)^{-1/2}b^*)/((GG^T)^{1/2}b^*)\), where \(b^*\) is the right eigenvector of \(\lambda_\alpha((GG^T)^{-1/2}S^*(GG^T)^{-1/2})\) and where \(S^* = \arg \max_{S \in S} \lambda_\alpha((GG^T)^{-1/2}S(GG^T)^{-1/2})\).

We note that the above theorem provides an exact computation of the minimum condition number using a finite number of eigenanalyses.

As with the dominant eigenvalue ratio, iterative algorithms for determining the minimum condition number can be constructed. Also, there is a graph-theoretic interpretation for the optimal solution, albeit in terms of a graph related to \(GG^T\) rather than \(G\) itself. We omit these further analyses.

#### 3.2 A minimum eigenvalue-based measure

Let us consider maximization of the performance measure \(c_\lambda(KG) = \lambda_\alpha(KG)/\|KG\|_\infty\) with respect to the positive diagonal scaling \(K\), for symmetric positive definite \(G\). Using the optimization machinery illustrated in this paper—or more simply by using continuation-root-locus ideas—we can show that the optimizing gain
matrix is the one which makes the absolute-row-sums of $KG$ equal, and hence we can easily calculate the maximum value of the performance measure. The structure of the optimizing gain matrix is formalized in the following theorem.

**Theorem 9:** A gain matrix $K = K^*$ that maximizes the performance measure $c_A(KG) = \lambda_1(KG)/KG_\infty$ can be chosen as follows:

- $k_i^*$ arbitrary;
- $k_i^* = k_i^*(\sum_{j=1}^n |g_{ij}|)/(\sum_{j=1}^n |g_{ij}|)$, $2 \leq i \leq n$.

Once the optimizing gain matrix has been determined, we can trivially determine the optimal value of the performance measure $c_A(K^*G)$.

### 3.3 Diagonal Lyapunov function design

We are also interested in maximizing Lyapunov exponent-based performance measures, such as $\tilde{p}(KG) = \sqrt{\lambda_1((KG)^T P + PG)}/\lambda_0(P)||K||_\infty$ with respect to the positive diagonal gain $K$ and the diagonal Lyapunov function $P$. It is straightforward to show that this optimization can be carried out by first optimizing $\lambda_1(G^T P + PG)/\lambda_0(P)$ with respect to positive diagonal $P$; this optimizing Lyapunov function in turn permits us to obtain a set of gain matrices and corresponding Lyapunov functions for which the original performance measure is maximized. The significant step of optimizing $\lambda_1(G^T P + PG)/\lambda_0(P)$ with respect to the diagonal matrix $P$ can be carried out using the optimization approach illustrated in this paper. In fact, this problem of optimal diagonal Lyapunov function design has received considerable attention of its own right, see e.g., Khalil (1982), Geromel (1985) and Hershkovitz (1992) and our design approach provides new structural insight in this realm. For this reason, and because the result is relatively intricate, we have addressed the diagonal Lyapunov function design problem in a separate paper (Roy and Saberi 2006), for arbitrary square $G$. We stress that the methodology for diagonal Lyapunov function design is an example of the scaling-design methodology illustrated in this paper.

### 3.4 A foray into eigenvalue placement

Let us consider the problem of placing the eigenvalues of $KG$ at desired locations by choosing the diagonal gain matrix $K$. Classically, the eigenvalue placement problem is significant for performance and robustness analysis in control systems. Eigenvalue placement is well-understood in the centralized case (i.e., for dense $K$). Here, we give some first steps toward understanding eigenvalue placement in the decentralized case, i.e., using scaling, for positive definite symmetric $G$. In studying eigenvalue placement through scaling, it is natural (as with all the other design problems) to consider ratios of eigenvalues rather than the eigenvalues themselves. We find it most convenient, in particular, to consider design of the eigenvalue ratios $\lambda_{i1}(KG), \ldots, \lambda_{in-1}(KG)$, $\lambda_{1}(KG), \ldots, \lambda_{n}(KG)$ are the eigenvalues of $KG$ in order from smallest to largest. In analogy with the dominant eigenvalue ratio, it can be shown that any subset of these eigenvalue ratios can be made arbitrarily large using diagonal scaling, while (unlike in a centralized setting) the eigenvalues cannot be made arbitrarily close. The interesting question then is how close the eigenvalues can be placed, i.e., how small the eigenvalue ratios can be made.

We apply the optimization methodology developed in this paper to obtain two partial results on how small the eigenvalue ratios can be made.

(i) Given a particular diagonal scaling $K$ and corresponding eigenvalue ratios $\lambda_{i1}(KG), \ldots, \lambda_{n}(KG)$, we develop a test for whether the diagonal scaling $K$ can be perturbed in such a manner that all eigenvalue ratios improve (decrease).

(ii) We develop an iterative (gradient search-type) algorithm using which a particular eigenvalue ratio $\lambda_{i+1}(KG)$ can be minimized, while the eigenvalue ratios $\lambda_{i-1}(KG), \ldots, \lambda_{21}(KG)$ are held fixed.

Let us begin with a test for whether a diagonal scaling can be improved through perturbation.

**Theorem 10:** Consider a particular diagonal scaling $K$ and the corresponding eigenvalue ratios $\lambda_{i1}(KG), \ldots, \lambda_{n}(KG)$. Then there exists a diagonal matrix $\Delta$ such that $\lambda_{i+1}(K + \epsilon \Delta G) < \lambda_{i+1}(KG)$ for all $i \in \{1, \ldots, n-1\}$ and all $0 < \epsilon < \delta$, some $\delta$, if the matrix

\[
S \triangleq \begin{bmatrix}
v_{11}^2 - v_{21}^2 & \cdots & v_{1n}^2 - v_{2n}^2 \\
\vdots & \ddots & \vdots \\
v_{n1}^2 - v_{n2}^2 & \cdots & v_{nn}^2 - v_{n,n}^2
\end{bmatrix}
\]

(where $v_{ij}$ is the $j$th element of the $i$th eigenvector) has at least one strictly positive vector in its range space.

**Proof:** We can prove the theorem by showing that there exists a $\Delta$ such that the sensitivities of eigenvalue ratios $\lambda_{i1}(KG), \ldots, \lambda_{n}(KG)$ to a perturbation $\Delta$ (i.e., the derivatives of $\lambda_{i+1}(K + \epsilon \Delta G)$ with respect to $\epsilon$, evaluated at $\epsilon = 0$) are all negative. Using standard eigenvalue sensitivity arguments, one can easily show...
that the sensitivities of the \( n - 1 \) eigenvalue ratios are given by \(-\Delta S\Delta\), where
\[
\Lambda = \begin{bmatrix}
\hat{\lambda}_{21}(KG) & & \\
& \ddots & \\
& & \hat{\lambda}_{n,n-1}(KG)
\end{bmatrix}
\]
and \( \tilde{\Lambda} \) is a column vector which contains the diagonal entries of \( \Lambda \). Noticing that \( \tilde{\Lambda} \) is strictly positive, we notice that there exists \( \Lambda \) such that the sensitivities are all negative if \( S \) has a strictly positive vector in its range space.

Several notes about this theorem are needed.

(i) The existence of a strictly positive vector in the range space of \( S \) can be checked using standard linear algebraic techniques.
(ii) The condition can equivalently be phrased in terms of non-existence of positive vectors in the null space of \( S^T \).
(iii) Although we chose to consider all sequential eigenvalue ratios in the theorem, we can analogously study other sets of eigenvalue ratios. When we only consider the single eigenvalue ratio \( \hat{\lambda}_m/\hat{\lambda}_l \) this analysis reduces to the analysis given in Theorem 3.
(iv) We note that the check for improvement is a local one: our analysis gives no indication of how a particular eigenvalue ratio will change due to a large change in \( K \).
(v) The condition above is not quite necessary and sufficient because a particular eigenvalue ratio may have sensitivity 0 and yet decrease (in a second- or higher-order sense) in response to a perturbation. However, if we limit ourselves to checking for perturbations that improve all eigenvalue ratios in a first-order sense, then the condition is both necessary and sufficient. We note that various necessary conditions for improvement of the eigenvalue ratios can also be developed using similar arguments; we do not pursue these analyses further here in the interest of space.

Finally, let us develop an algorithm using which a particular eigenvalue ratio \( \hat{\lambda}_{i+1,l}(KG) \) can be driven to a local minimum or inflection point, while the eigenvalue ratios \( \hat{\lambda}_{i,j-1}(KG), \ldots, \hat{\lambda}_{2,l}(KG) \) are held fixed. This iterative algorithm is based on the same sensitivity argument used to prove Theorem 10: by perturbing the diagonal scaling \( K \) appropriately, we can modify and hence minimize \( \hat{\lambda}_{i+1,l}(KG) \), while leaving the other eigenvalue ratios (essentially) unchanged.

**Algorithm:**

**Initialization:** Consider a gain matrix \( K \) for which \( \hat{\lambda}_{i,j-1}(KG), \ldots, \hat{\lambda}_{2,l}(KG) \) need to be held fixed while \( \hat{\lambda}_{i+1,l} \) is decreased. Initialize an iteration gain matrix as \( K_1 = K \). Also, choose an iteration stepsize \( \epsilon \) and a tolerance \( \delta \). Initialize a counter variable \( j \) as 1.

**Recursion:** Repeat

(i) increment \( j \) by 1;
(ii) find the eigenvectors \( v_1, \ldots, v_{i+1} \) of \( KG \);
(iii) compute a basis for the nullspace of
\[
Q = \begin{bmatrix}
v_{11}^2 - v_{21}^2 & \cdots & v_{i-1,1}^2 - v_{i+1,1}^2 \\
\vdots & \ddots & \vdots \\
v_{1n}^2 - v_{2n}^2 & \cdots & v_{i-1,n}^2 - v_{i+1,n}^2
\end{bmatrix};
\]
(iv) using this basis, find the projection of the vector
\[
\begin{bmatrix}
2v_{11}^2 - v_{21}^2 - v_{i-1,1}^2 + v_{i+1,1}^2 \\
\vdots \\
2v_{1n}^2 - v_{2n}^2 - v_{i-1,n}^2 + v_{i+1,n}^2
\end{bmatrix}
\]
into this nullspace. Call the projection \( \tilde{k} \); 
(v) Compute the iteration gain matrix \( K_j = K_{j-1} + \epsilon \text{diag} \tilde{k} \).

Until \( \|\tilde{k}\| < \delta \).

**Termination:** The desired gain matrix is \( K = K_f \).

It is straightforward to formalize that the above algorithm decreases the ratio \( \hat{\lambda}_{i+1,l} \) while the other eigenvalues are essentially held unchanged (i.e., their change over the duration of the algorithm can be made arbitrarily small when a sufficiently small time-step is used. This formalization does not significantly add to understanding of the eigenvalue-placement problem, so we omit it.

**Example:** Let us conclude our discussion of eigenvalue placement by illustrating the iterative algorithm for designing eigenvalue ratios. In particular, consider the
\[
topology matrix \ G = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 2
\end{bmatrix}
\]
and
\[
initial scaling \ K = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3.3
\end{bmatrix}.
\]
Meanwhile, the eigenvalue ratio \( \lambda_2/\lambda_1 \) (thin line) and \( \lambda_3/\lambda_2 \) (thick line) are shown for the duration of the algorithm. We see that one eigenvalue ratio is held constant while the other is decreased to a minimum.

For this scaling, we find that \( K_G \) has the following eigenvalue ratios: \( \lambda_2/\lambda_1 = 3.00 \) and \( \lambda_3/\lambda_2 = 1.616 \). Using Theorem 10, we can check that both eigenvalue ratios can be improved through perturbation of \( K \). Our aim is to modify the scaling \( K \) so as to keep the eigenvalue ratio \( \lambda_2/\lambda_1 \) fixed at 3.00, while minimizing \( \lambda_3/\lambda_2 \). Applying the iterative algorithm described above with time step \( \epsilon = 1E-5 \), we find that the eigenvalue ratio \( \lambda_3/\lambda_2 \) can be reduced to 1.33, by using the scaling

\[
K = \begin{bmatrix}
2.55 & 0 & 0 \\
0 & 1.14 & 0 \\
0 & 0 & 2.60
\end{bmatrix}.
\]

Meanwhile, the eigenvalue ratio \( \lambda_2/\lambda_1 \) remains constant to five significant digits.

The evolution of \( \lambda_2/\lambda_1 \) and \( \lambda_3/\lambda_2 \) are shown for the duration of the iterative algorithm in figure 1.

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